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MAGIC SQUARES OF ORDER FOUR

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[Plate 1]

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A brief history of work on the 4×4 magic square is presented, with particular reference to Frénicle's achievement over 300 years ago of establishing 880 as the number of essentially different squares by using the method of exhaustion (not convincingly repeated except by computer in 1976). He also established several central theorems. Our paper confirms the number 880 by a wholly new method of 'Frénicle quads' and 'part sums', which leads to the classification of all solutions into, initially, six 'genera' one of which has no members and thence to the enumeration of all possible solutions by analytical methods only. The working leads also to the first analytical proof independent of solutions that 12 and only 12 patterns formed by linking 'complementary' numbers within a square are necessary and sufficient to describe all solutions – a fact which has been known since 1908, but not hitherto proved. A second method of construction and partial proof, greatly shortened by what has gone before, is also described. This yields a highly symmetrical list of the 880 magic squares. Together the two methods combine to explain many of the special characteristics and otherwise mysterious properties of these fascinating squares. The complete symmetrical list of squares ends the paper.

1. THE MAGIC SQUARE OF ORDER FOUR AND ITS HISTORY

A magic square consists of integers arranged in the form of a square so that the sum of these integers in every row, in every column and in each of the two principal diagonals is the same. Any square can be subjected to a reflexion and/or rotations through 90° without losing its magic character. Squares are said to be *essentially different* if they cannot be transformed into one another by rotations and/or reflexion. Every essentially different square can thus be written in eight forms and a set of essentially different squares can be assembled in a great variety of ways. If the integers forming a magic square are the consecutive positive numbers from 1 to n^2 inclusive, the square is said to be 'normal' and of the nth order. The sum of the numbers on every row, column and the two principal diagonals is then easily seen to be $\frac{1}{2}n(n^2+1)$.

Over three hundred years ago Bernard Frénicle de Bessey (1602–1675) established that there are 880 and only 880 essentially different normal 4×4 magic squares. Bernard Frénicle was 'conseiller du Roi en sa cour des monnaies' and spent his spare time in research in numbers. He was thus in the strict sense an amateur mathematician of distinction working in Paris during the great period of French mathematics of the seventeenth century. He resurrected the long-known but often rather despised 'method of exhaustion' to solve difficult arithmetical problems. By considering general conditions for the 4×4 magic squares and excluding those arrangements which cannot comply with these conditions, he progressively arrived at all essentially different solutions. Frénicle's list of solutions was first published posthumously in 1693 as an integral part of a substantial treatise entitled *Des Quarrez ou Tables Magiques*, one of four treatises collected together by Phillipe de la Hire (1640–1718), a French geometer who also interested himself in the construction of magic squares. The treatise was republished in *Les Mémoires de l'Académie des Sciences* in 1731 in the Hague* and it is this publication which is the more readily accessible today.

Several of the properties of 4×4 magic squares are general and hold when numbers other than 1–16 are used. It is however convenient here, except where otherwise stated, to speak only of *normal squares* constructed from the numbers 1–16, or, alternatively, of the consecutive numbers 0–15, so that the sum of the integers in the rows, columns and principal diagonals is 34 or 30 respectively.

There is a huge corpus of recorded work on magic squares spanning the centuries since at least 2200 B.C. when the only 'fundamental' 3×3 magic square using the numbers 1–9 namely

2	9	4
7	5	3
6	1	8

which is known as the *lo-shu*, is said to have been brought to man by a turtle from the river Lo in the days of the legendary Emperor Yu of China (Boyer 1968). A Jaina inscription of the twelfth or thirteenth century giving the 4×4 magic square

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

was reported in 1904 as having been found in Khajurado, India. This square is said to be *pandiagonal* or *Nasik*, sometimes *diabolical*. It has the property that not only do the numbers in the rows, columns and principal diagonals add to 34, but so also do the numbers in the 'short broken diagonals', namely, 2 12 5 15, 1 11 6 16 and those in the 'long broken diagonals', namely, 7 6 10 11, 14 2 3 15, 4 16 13 1, 9 12 8 5. Pandiagonal squares are 'continuous'. They have the property that, if they are extended indefinitely by repetition or if wrapped round a cylinder or if drawn on a torus, any square block of sixteen numbers so formed still remains a pandiagonal magic square. If the short broken diagonals of a square add to 34, but not the long broken diagonals, then the square is said to be *semi-pandiagonal* or *semi-Nasik*.

* It is a cause of satisfaction for us that our paper was completed just 250 years after this important event.

A magic square is said to be *symmetrical* if each of those pairs of numbers which are symmetrical in the centre point of the square add to half the sum of the numbers in each row, column and principal diagonal – here 17. The magic square shown below, seen in the famous

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

engraving entitled *Melancholia* by Albrecht Dürer (1471–1528), plate 1, is symmetrical. Two impressions of this engraving are in the British Museum. The square contains the deliberate pun of the date of the engraving, 1514, appearing at the centre of the bottom row. From the list of solutions in this paper it is easily checked that there are 32 essentially different magic squares which can be written with the numbers 15, 14 in these positions, but only four of them, of which Dürer's is one, are symmetrical.

Another particular group of normal 4×4 magic squares will be shown in §6 to give the essentially different solutions to the well-known problem of placing the sixteen court cards from a pack in the form of a square so that no row, no column and neither of the two principal diagonals contains more than one card of each suit and one card of each rank. In early editions of his famous book *Mathematical recreations and essays* W. W. Rouse Ball (1850–1925) stated that the 'magic card problem is easily solved' and gave one example. Later editions, including a posthumous edition published in 1944 revised by H. S. M. Coxeter, stated *incorrectly* that the number of essentially different solutions is 72, an error eliminated in the 1974 revised edition. An easy independent proof that the correct answer is 144 is given in §6 and the actual solutions can be quickly identified in the list of all solutions.

To complete the definitions, we call any two numbers in a square which add to 17, i.e. to half the sum of the numbers in the rows, columns and principal diagonals, *complements*. Any two squares are called complements if they can be obtained from one another by replacing each number in one of the squares by its complement. The complement of a magic square is also magic, and the complements of pandiagonal, semi-pandiagonal, symmetrical and 'magic card' squares are also pandiagonal, semi-pandiagonal, symmetrical and 'magic card' squares respectively, although not necessarily essentially different.

Before Frénicle's time it had been thought that there were only 16 essentially different normal 4×4 magic squares. His method of constructing his list, as well as giving a set of 880 solutions, also established that these are the only essentially different solutions. Work on magic squares in the early 1900s and in the 1930s indicate a surprising ignorance of Frénicle's treatise and even sometimes of the existence of his definitive list. There are an astonishing number of errors in the extensive literature, many of which would not have occurred had correct 'counts' been made from his list of those squares with this or that special characteristic. In particular, solutions which Frénicle classified as α fulfil the necessary and sufficient conditions for a square to be pandiagonal and those which he classified as β, γ together define all semi-pandiagonal squares. Frénicle stated correctly that there are 48 solutions with his classification α, 192 with classification β and 192 with classification γ. The 48 pandiagonal and 384 semi-pandiagonal squares have thus been in print since 1693 and readily accessible since 1731,

Albrecht Dürer's 'Melancholia' (The British Museum). Note the four-by-four magic square in the upper right-hand corner in which the date 1514 appears in the two middle cells of the bottom row.

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although incorrect statements about the number of pandiagonal squares appeared in published literature as recently as 1933 (Lehmer 1933) subsequently corrected (Rosser & Walker 1938).

Frénicle's method, leading as it does to a list which lacks symmetries and pattern, does not lend itself to accurate counting of other interesting characteristics, as an example from the published treatise of 1731 itself shows. Frénicle first found all solutions with the number 1 in an outside corner; then all those with 2 in an outside corner (no solution can have both 1 and 2 or both 1 and 3 in an outside corner); then all those with successively 3, 4, 5, 6, 7, as the *smallest* number in an outside corner. This completed the search since, as will be shown later, Frénicle proved that the numbers in the four corners must add to 34. The resulting table is given on the first page (p. 368) of the relevant section of the 1731 text and appears as below:

1.	208	208
2.	200	200
3.	204	166
4.	238	178
5.	216	64
6.	206	48
7.	230	16
	Somme	880

The middle column purports to show all solutions which contain the corresponding number in the left-hand column as an outside corner of the square. The right-hand column shows the number of all solutions for which the corresponding number in the left-hand column is the *smallest* number in any of the four outside corners. The important right-hand column and the resulting total number of solutions is correct. The less important middle column contains two errors – the correct counts for those squares with 3 and with 6 in an outside corner are respectively 202 and 228, not 204 and 206. These counts are easily verified in the symmetrical list of solutions given in this paper, but are tiresome to verify from Frénicle's unsymmetrical 'sequential' list.

More important than conflicting enumerations of subgroups of solutions is that not all authors in the intervening years since 1693 have appreciated or accepted that Frénicle's method constitutes *proof* that there are only the 880 essentially different solutions – admittedly proof which could be verified in pre-computer days only by repeating the whole arduous process of step-by-step calculation which Frénicle almost certainly used, the task of checking for possible omissions being as wearisome and requiring as much skill as the original working. W. S. Andrews (1847–1929), in his considerable text *Magic squares and cubes*, first published in Chicago in 1908 and still much borrowed from public libraries in this edition, stated categorically that the number 'of diverse squares has been estimated by different writers as 880 ... it can however be easily proven than no less than 4352 may be constructed ...'. By using Frénicle's rules (§2(*b*)) it is a simple matter today to program a computer to produce all possible solutions – using, essentially, Frénicle's own method of exhaustion. In 1976 W. H. Benson and O. Jacoby fed such a program into an IBM 1130 computer in Pennsylvania. The resulting total of different solutions was 880, confirming Frénicle's working. A computer print-out of these solutions is reproduced (in minuscule print) at the end of their book, *New recreations with magic squares* (1977).

In a final chapter of his 1908 book, Andrews quotes a contemporary, L. S. Frierson, as showing that there are 'at least eleven different plans which can be used in connection with 4 × 4 squares ... which clearly differentiate the various types of squares'. These are the *patterns* made by lines joining the eight pairs of complementary numbers within the square. There are in fact twelve such patterns, one of which can be simply derived from the eleventh of those Frierson depicted, but which he inexplicably missed. These twelve patterns are not only necessary to describe the 880 solutions but sufficient.

In 1910, H. E. Dudeney (1857–1931) published in *The Queen* of 15th January a definitive article on the 4 × 4 magic square, in which he describes how to construct all 880 *with the help of the twelve patterns*, ending by saying, 'I have thus provided the reader with a new and simple method of writing out the whole of the 880 primitive solutions to this famous problem. More than this it is not possible to obtain'. *The Queen*, which at that time described itself as 'The Lady's Newspaper', regularly published articles featuring mathematical puzzles edited by E. Bergholt at the turn of the century much as the *Scientific American* does today. Dudeney begins his article: 'Bernard de Frénicle, between the years 1666 and 1699 [*sic*], investigated the subject, and in a book published in Paris in 1729 [*sic*] gave a complete list of 880 such different squares, which he declared to be all that exist. In 1886 these figures were confirmed independently by Frolow and Delannoy ... So far as I know, the list has never been printed in this country. The recent appearance in America of a book on magic squares, in which a writer disputes Frénicle's results, has led me to investigate the matter anew ...'. Dudeney was clearly referring to Andrews's book (which some modern references give incorrectly as appearing first in 1917) in which, as has been said, eleven of the twelve patterns had been given. Neither in this article nor in his subsequent book *Amusements in Mathematics* of 1917 did Dudeney prove (or specifically claim to prove, although he seems to imply this in *The Queen* as quoted above) that there were only 880 solutions or that the twelve patterns are sufficient to describe *all solutions however many*, but merely gave a method of construction of the 880 solutions which Frénicle (who had died in 1675) had shown to be the only solutions. Dudeney's method depended on there being only the 12 patterns without this ever having been proved unless by checking against Frénicle's own list.

In May 1910, Bergholt published in *Nature* 'A new and completely general formula' for the construction of magic squares of sixteen cells, giving as reference the re-publication of Frénicle's treatise in The Hague in 1731. This general form, reproducted by Ball, is the basis of most methods of construction of 4 × 4 magic squares (not only normal magic squares formed by the numbers 1 to 16) used by subsequent authors, in particular by Benson & Jacoby (1976). M. Kraitchik in his *Mathematical Recreations*, published in America in 1943 and based on an earlier book published in Brussels in 1930 and not now easily available, shows how to construct the 432 pandiagonal and semi-pandiagonal squares and states the number of solutions in each of five other categories which he defines, making in all the total of 880. There are over 400 separate articles on magic squares listed in W. L. Schaaf's *Bibliography of recreational mathematics* (1970, 1973, 1978). It would be a practical impossibility, and probably not particularly profitable, to attempt to follow up so many references. Only those articles and books which we have actually seen and which can be readily seen by other workers in the United Kingdom are referred to here, the most recent of these being the 1976 book by Benson & Jacoby.

There is no great difficulty in *constructing* the 880 solutions by one or other of the several now well-known methods, nor in proving from elementary considerations that there are 48 pandiagonal and 384 semi-pandiagonal solutions, nor indeed in constructing the 864 solutions not deemed 'irregular'. The problem arises of finding an analytical proof that there are 16 and only 16 essentially different 'irregular' solutions and in understanding why, as some irregular solutions exist, there should be only 16. Hitherto it would seem that the only proofs have been Frénicle's original proof by exhaustion and the similar computer proof undertaken by Benson & Jacoby in 1976. In this paper, as well as giving two separate methods of constructing the 880 solutions, one of which leads to the symmetrical list, we give a completely independent analytical proof of Frénicle's result. The method leads incidentally to a proof (which may be the first in existence which *is independent of actual solutions*) that the twelve patterns already mentioned are necessary and sufficient to describe all solutions.

Our interest in this fascinating age-old problem was aroused in the first instance by a self-imposed problem concerning the well-known Fifteen Puzzle or 'Boss'. The boss is a toy which enjoyed great popularity in the last century in which the numbers 1–15 are arranged in a 4×4 square box with the number 16 represented by a space. In the 'normal boss' the initial arrangement is that of the 'normal array'

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	(16)

with the number 16 missing. This normal array will be much used in what follows. The boss puzzle is to arrive at any other pre-determined arrangement of the fifteen numbers when they can be moved around only vertically or horizontally. The puzzle is fully analysed by Ball. Half and only half of all possible arrangements can be achieved. The question arose whether all magic squares, or if not all then how many, could be achieved with the normal boss, the space being assumed filled by the number 16. An obvious hypothesis, which proves to be incorrect, is that exactly half could be achieved. This required a list of all magic squares, or at least a way of constructing such a list, with the greatest possible degree of symmetry.

One simple method seemed to be to work with the numbers 0 to 15 expressed in scale 4, so that every solution must be a combination of two 4×4 matrices each of which is made up of four each of the digits 0, 1, 2, 3 in some arrangement. Each cell of a completed square will then contain two digits from among the sixteen different ordered pairings

00	01	02	03
10	11	12	13
20	21	22	23
30	31	32	33

each of these pairings occurring once and only once. These two arrays are equivalent, the second being obtained from the first by subtracting 1 from each of the numbers 1–16 and expressing the resulting numbers 0–15 in scale 4. There is nothing new in this method of presentation. Indeed, Ball uses it to describe several particular squares of order 4 and of orders greater than 4. It forms the basis of our method used in constructing the list of solutions by the 'matrix method'.

In the narrow context of finding all solutions to the magic card problem, this 'matrix method' led directly to the (correct) answer 144 as will be shown in §6. That this was not the accepted answer as given by Ball was an irresistible incentive to continue with the method to produce a complete list for its own sake (and usefulness in making other checks and discoveries); particularly when, at a much later stage of our work, it became apparent that no symmetrical list seemed to exist and that even Frénicle's (sequential) list of the 880 solutions is available in the United Kingdom only in the British Library. The full list of solutions in 00–33 notation given here, which exhibits their many remarkable symmetries, has been the firm foundation on which all subsequent work in this paper was based and from which the analytical proof was developed.

We have had at our disposal two very different methods, each illuminating in its own way, to arrive at a proof of Frénicle's result: the 'matrix method' and the method we have come to call the 'Frénicle-quad and part-sum method'. After initial rapid progress each method, in its turn, brought us up against the fundamental difficulty of establishing why there are 16 and only 16 irregular solutions. However, the Frénicle-quad method proved much the more powerful and has in addition certain important simplicities which give it a marked advantage in enumerating the total number of solutions. In contrast, the matrix method, with its inherent symmetries, is better for explaining certain facts (as well as in solving the 'boss' problem which would not have yielded easily otherwise) – facts which might not have been understood without matrices. In combination the two methods give the best of both worlds.

We have not tried to resist recording a number of the fascinating facts relating to these magic squares which have emerged as the work has progressed. Many if not all will be known to others, but each in turn was a new discovery for us and added to the delight and excitement of seeking answers, as others have done through the centuries, to the many mysteries in the behaviour of these first 16 positive integers when arranging themselves into what are truly called *magic* squares.

2. The groundwork

(*a*) *Frénicle's square*

Frénicle established in elegant prose a set of conditions which hold for all 4×4 magic squares and on which he based the logical deduction of his list of solutions by the method of exhaustion. Out of sentiment and respect for his great achievement we use his lettering throughout for the elements of any square forming a solution (the letter *j* is not used), namely, where the letters

$$\begin{matrix} a & b & c & d \\ e & f & g & h \\ i & k & l & m \\ n & o & p & q \end{matrix}$$

represent the numbers 1–16 in some arrangement and so form eight *complementary pairs* of numbers which add to 17. By definition the numbers in the four rows, the four columns and the two principal diagonals add to 34. We define as a '*quad*' any set of four numbers chosen from 1 to 16 which add to 34, the order of the elements in the quad being irrelevant, and we call any two, three of four quads '*compatible*' if they do not duplicate any number from 1 to 16. Any set of *four compatible quads*, which by the above definition must contain each of the numbers 1–16 once and once only, we define as a *quadset*, the order of the quads in a quadset being irrelevant. The rows of a magic square thus form a quadset and the columns another; so also do the two principal and two short broken diagonals of pandiagonal and semi-pandiagonal magic squares, and the four long broken diagonals of pandiagonal squares.

(*b*) *Frénicle's rules*

We give below Frénicle's proofs that (i) the sets of four numbers (quads) $a\,d\,n\,q$, $f\,g\,k\,l$, $b\,c\,o\,p$, $e\,i\,h\,m$ in the magic square as depicted above (quads which we shall call, respectively, the *corner*, *centre*, *vertical* and *horizontal quads*) also have the sum 34 and, being by definition compatible, thus form a quadset which we call the *Frénicle quadset*; (ii) the sets of four numbers in opposing 'petits quarrez des angles' which we call *Frénicle's corner squares* always have equal totals, that is, $a+b+e+f = l+m+p+q$ and $c+d+g+h = i+k+n+o$, but these totals are not necessarily 34; (iii) the sets of four numbers in opposing 'quarrez de trois' which we call *Frénicle's extended corner squares* likewise always have equal totals, that is $a+c+i+l = f+h+o+q$ and $b+d+k+m = e+g+n+p$, but these totals are not necessarily 34; (iv) from this, if the numbers in any one of the corner or extended corner squares add to 34, then the numbers in all eight add respectively to 34, and so the corner squares *and* the extended corners squares (both of which then form compatible sets of four numbers adding to 34) form quadsets.†

Frénicle's proofs of his theorems are direct. First suppose that the sum of the four corners is less than the required 34, so that $a+d+n+q = 34-\epsilon$. Since the top row and bottom row each add to 34 and thus their sum is 68, the other four elements involved must satisfy $b+c+o+p = 34+\epsilon$. Similarly from the first and last columns $e+i+h+m = 34+\epsilon$. On considering now the sum of the two central rows (or columns), it follows that the central set $f+g+k+l = 34-\epsilon$. But the two principal diagonals each add to 34, and thus the sum of the corners and the central square, just found to be $68-2\epsilon$, must be 68. Thus ϵ cannot be positive, nor, from the same argument, can it be negative and (i) is proved. To establish conditions (ii), (iii), he notices that both $a+b+e+f$ and $l+m+p+q$ when added respectively to $c+d+g+h$ total 68 and they are therefore equal. Similarly the numbers in the other two opposing corner squares have equal sums. Again, $a+c+l+i$ and $f+h+o+p$ when added respectively to $e+g+n+p$ total 68 and are thus equal; as are similarly the other opposing 'quarrez de trois', namely $b+d+k+m$ and $e+g+n+p$. If the numbers in one corner square or in one extended corner square add to 34, since the numbers in any two rows or in any two columns add to 68, it follows immediately that the numbers in all corner squares and all extended corners squares add respectively to 34. The full condition (iv) is then easy to establish.‡

From (i) above, a set of ten *Frénicle equalities* immediately follow, namely $a+d = o+p$, $b+c = n+q$; $e+h = k+l, f+g = i+m$; $a+n = h+m, e+i = d+q$; $b+o = g+l, f+k = c+p$;

† Surprisingly, Kraitnik (1938) states that these totals are always 34, whereas they are 34 only when the magic square is pandiagonal or semi-pandiagonal.

‡ Frénicle also made use of two other pairs of subsidiary 2×2 squares within the magic square: those formed by the letters $b\,c\,f\,g$, $k\,l\,o\,p$, $e\,f\,i\,k$, $g\,h\,l\,m$. He uses them to describe his classifications α, β, γ, δ and 'unmarked' which are mentioned in §1 but which are not again referred to as they do not contribute to either of the methods used here to prove Frénicle's result.

$a+q = g+k$, $f+l = d+n$. When the equalities (iv) obtain, that is, when $a+b+e+f = c+d+g+h = i+k+n+o = l+m+p+q = 34$, then $b+e = 34-a-f = l+q = 34-m-p$, and $c+h = 34-d-g = k+n = 34-i-o$; so that the elements of the short broken diagonals add to 34 and the solution, by definition, is either pandiagonal or semi-pandiagonal.

The Frénicle quads and the quadsets they define are of particular significance and importance in all which follows. We need however to discuss further the properties of *all quads* and of *all quadsets* in general.

(*c*) *Quads*

There are 86 ways in which four numbers adding to 34 can be chosen from the numbers 1–16, and thus 86 quads. They are listed in the different 1–16 and 00–33 notations in the two halves of Appendix I for convenience and ready reference. There are three distinct groups of quads. The first group consists of the 28 *self-complementary quads*, that is quads in which two pairs of numbers within the quad add to 17. The second and third groups consist of pairs of *mutually-complementary quads*, that is pairs of quads such that each number in one quad has its complement in the other. The distinction between the second and the third groups (of 12 and of 17 pairs of mutually-complementary quads respectively) is of great significance as will become apparent later: a first and immediate distinction being that, whereas *compatible pairs* can be found to form a quadset from within the second group, no pair of mutually-complementary quads from the third group can form a quadset in conjunction with *any other mutually-complementary pair* whether from the second or third groups. This remarkable and important distinction can be easily checked from the list of all quads. A further distinguishing characteristic (used in explaining the matrix method of proof) is that whereas the 28 self-complementary quads and the 2×12 mutually-complementary quads of the second group when expressed in the 00–33 notation are each composed of four elements, the digits of which in *both* the radix *and* the unit positions are either 0 1 2 3 or 0 0 3 3 or 1 1 2 2 respectively in some order, this property does *not* hold for any quad in the third group. We shall show that no quad in the third group can occur in any solution as a row, or as a column, or as a Frénicle quad (although all occur somewhere in the full list of solutions as principal diagonals).

(*d*) *Quadsets*

A quadset may contain (*a*) at least three three self-complementary quads, then evidently the fourth has also to be self complementary; (*b*) two and only two self-complementary quads, then the other two must necessarily be a pair of mutually-complementary quads; (*c*) no self-complementary quads but at least one pair of mutually-complementary quads, then it must consist of two pairs of mutually-complementary quads; (*d*) one and only one self-complementary quad and no pair of mutually-complementary quads, in which case the three non-self-complementary quads can be said to have 'no complementarity'; (*e*) no self-complementary quad and no pair of mutually-complementary quads, in which case the four quads have no complementarity anywhere. We notice that quadsets which meet the conditions (*a*), (*b*), (*c*) are themselves self complementary, whereas for any set in (*d*), (*e*) there will exist an essentially different complementary set within the same respective classification.

(*e*) *Part sums*

We define as a *part sum* the sum of any two numbers within a quad whenever this sum is not greater than half the sum of the numbers in the quad, i.e. whenever here the sum is not greater than 17. There are thus three different part sums in every quad, say α, β, γ, (which we define as a *triad*), with their 'supplements' $34-\alpha$, $34-\beta$, $34-\gamma$ occurring in each case as sums of the remaining elements. If a part sum equals 17 then so does another, and the quad is self complementary. Otherwise we can pair a quad with its complement (as earlier defined) to give a pair of mutually-complementary quads. In particular, the triads of part sums for two mutually-complementary quads are identical.

(*f*) *Parities within quads*

We establish first some simple properties concerning the parities of the elements in all quads forming quadsets. First we notice that *no two quads in a quadset can be composed of all odd or all even elements*, for the sum of all odd numbers $1+3 \ldots +15 = 2\times 32$ and the elements of any two quads together add to 68. Moreover, since the elements in any quad add to 34, the number of odd elements must be even so a quadset containing just one quad with elements of the same parity cannot exist. For consider the set of four triads forming the part sums for a Frénicle quadset. An all-odd as well as an all-even Frénicle quad would have all three part sums even. Two all-even triads are then impossible. For, if two triads have all-even part sums so that the part sum model can be written as (2 2 2) (2 2 2) (1 1 2) (1 1 2) where here 1, 2 represent any odd, even number respectively, the required links to give necessary Frénicle equalities within a solution, as explained in § 3 (*a*), cannot be formed. It follows, importantly, that

(i) *no Frénicle quad can have elements which are all even or all odd*, and hence, *since their sum is even* (34), *two must be even and two odd*, and

(ii) *triads of part sums of Frénicle quads must be such that two part sums are odd and one is even.*

(*g*) *Quads with a common part sum*

A principal step in our later arguments is to prove that the Frénicle quadset of any magic square always possesses a part sum common to all four quads. We find also that all rows and all columns (and all diagonal quadsets in pandiagonal and semi-pandiagonal solutions and long broken diagonal quadsets in pandiagonal solutions) also have respectively a common part sum. It is convenient to discuss here the severe restrictions imposed by the existence of such a common part sum *on each quad of a quadset*. We find that if this is less than 17, only the values 9, 13, 15, 16 are allowed and for each there are four and only four number pairs adding to the required total that can be used in the construction (to be called 'constructive' pairs), and thus each of them must be used, one for each quad, the same holding for their four 'supplements'. If the common part sum is 17 (so that every quad is self complementary) there are eight such number pairs, two of which must be used in each quad.

To prove this statement call the common part sum s and consider the possibilities when $s < 17$ (when $s = 17$ there is nothing to prove). For $s < 17$ to be a common part sum the quadset must contain four pairs of numbers adding to s (one pair for each of the four quads of the quadset) and four 'supplementary pairs' of numbers adding to $34-s$ to complete the quads, none of the sixteen numbers occurring more than once. If $s < 9$ there are fewer than four different sets of two numbers adding to s and no common part sum is possible. If $s = 9$ there are exactly four such pairs, namely those forming the constructive set given below. If $s = 10$, 11 or 12, there are at most five required partitions of s, at least two of which overlap with two of the supplementary pairs $34-s$, leaving at most only three compatible pairs of numbers to occupy four quads which is insufficient. For $s = 13$ with six suitable partitions, two overlap with $34-s$, leaving exactly the one constructive set given below. For $s = 14$ there are again six suitable partitions, but now three overlap with the partitions of $34-s$, leaving only three compatible pairs of numbers adding to 14 to occupy four quads. With $s = 15$ or 16, there are seven suitable partitions of s of which three overlap with partitions of $34-s$, leaving respectively the constructive sets given below. The statement is thus proved.

The constructive pairs of part sums

part sum 9	1	8,	2	7;	3	6,	4	5;
and supplement 25	16	9,	15	10;	14	11,	13	12;
part sum 13	1	12,	2	11;	3	10,	4	9;
and supplement 21	16	5,	15	6;	14	7,	13	8;
part sum 15	1	14,	5	10;	9	6,	13	2;
and supplement 19	16	3,	12	7;	8	11,	4	15;
part sum 16	1	15,	5	11;	9	7,	13	3;
and supplement 18	16	2,	12	6;	8	10,	4	14.

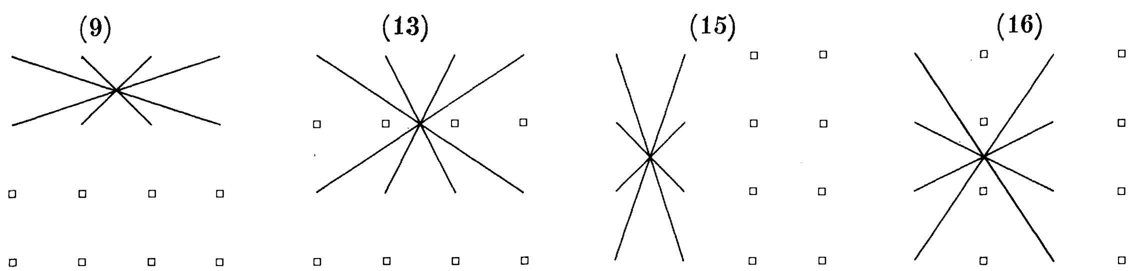
If these part sums are 'mapped' on the normal array

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

or, in the alternative form,

00	01	02	03
10	11	12	13
20	21	22	23
30	31	32	33

they form the symmetrical patterns shown below where the symbols □ also indicate positions on the array.



From this an easy check with the quad list confirms that all quads formed by *constructive pairs* of numbers which sum to 9, 13, 15, 16, or to 17 as defined above, together with pairs of numbers which are the supplements of these constructive pairs, are to be found in the second and first groups of quads in the quad list. In other words,

(i) *no quad in the third group of the quad list can be a component of a quadset for which the quads have a common part sum.*

Quads of the first group being self complementary necessarily contain two odd and two even elements. We thus have

(ii) *all quads of a quadset which have a common part sum, being from Groups 1 and 2 of the quad lists, contain two odd and two even numbers and so two of their part sums must necessarily be odd and one even.*

This condition is much more stringent than that proved above for Frénicle quads only, for some quads in the third group have two odd and two even elements but yet are not composed of constructive part sums and cannot therefore, from the above, belong to a quadset the quads of which have a common part sum.

(*h*) *Transformations* U, S, US

Before we turn to the main task of establishing the nature of all Frénicle quadsets, it is convenient first to consider certain standard transformations which leave unchanged the magic properties of any 4 × 4 magic square and also leave unchanged the Frénicle quadsets, the row quadsets, the column quadsets and the two principal diagonals.

If the two inner rows and the two inner columns of a magic square are interchanged as shown below, the transformation being labelled U, we have a square which is also magic.* The new square is moreover essentially different from the first. Similarly the square resulting from the transformation labelled S which effects interchanges between the numbers forming the diagonals of the four 2 × 2 corner squares of the main square as indicated (as it were turning the square inside out) is also magic and essentially different from either of the two magic squares already shown. It follows that these two transformations, when carried out in succession, result in a fourth magic square essentially different from the previous three. We thus have, associated with any 'lead solution' placed on the left, a set of four essentially different solutions of the form

					U				S				US		
a	*b*	*c*	*d*	*a*	*c*	*b*	*d*	*f*	*e*	*h*	*g*	*f*	*h*	*e*	*g*
e	*f*	*g*	*h*	*i*	*l*	*k*	*m*	*b*	*a*	*d*	*c*	*o*	*q*	*n*	*p*
i	*k*	*l*	*m*	*e*	*g*	*f*	*h*	*o*	*n*	*q*	*p*	*b*	*d*	*a*	*c*
n	*o*	*p*	*q*	*n*	*p*	*o*	*q*	*k*	*i*	*m*	*l*	*k*	*m*	*i*	*l*

Dudeney called the transformations U, S 'transpositions' and 'ruptures' respectively. Lehmer used the designations U, S, realizing (as had Dudeney some twenty years earlier) that this would have enabled Frénicle to reduce his list fourfold to 220 solutions. We note that U causes an interchange of the Frénicle corner and extended corner squares, leaving rows, columns, principal and short broken diagonals unchanged, but internally re-orientated; and S leaves all of these as well as the long broken diagonals intact, merely re-orientating them within the magic square.

(*i*) *Complements*

The complement of any magic square, that is, a square where every element is replaced by its complement is also magic. A square may be *self complementary* (if its complement is merely itself in a different orientation), or its complement may be obtained by the transformation U or S, or its complement may be an essentially different square not obtainable by these transformations.

* Curved lines are used throughout to indicate that whole rows/columns are interchanged while straight or angled lines are used when required to indicate interchanges of individual pairs of numbers or (with matrices later in this paper) of digits.

(*j*) *Reversals*

In certain circumstances, when a magic square is expressed in the 00 to 33 notation, the 'reversal' throughout of the digits in the radix and unit positions will also give a magic square, which, as with complements, may or may not lead by the transformations U, S, US to a new set of four essentially different magic squares. The effect of reversing the digits of any number 0–15 thus expressed in scale 4 is to give its reflexion in the diagonal running from the top left-hand to the bottom right-hand corner when all are arranged in the 'normal array' as shown below:

						reversal		
00	01	02	03		00	10	20	30
10	11	12	13		01	11	21	31
20	21	22	23		02	12	22	32
30	31	32	33		03	13	23	33

Expressed in the ordinary 1–16 notation for magic squares, this is the transformation represented by a single reflexion in the main diagonal thus:

						reversal		
1	2	3	4		1	5	9	13
5	6	7	8		2	6	10	14
9	10	11	12		3	7	11	15
13	14	15	16		4	8	12	16

The transformation effected by 'reversal', when written in ordinary 1–16 notation, namely

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	5	9	13	2	6	10	14	3	7	11	15	4	8	12	16

is not as easily 'seen' as being potentially effective. In practice the use of reversals proved a powerful method in producing the 00–33 symmetrical list of solutions, 256 essentially different solutions being automatically obtained in this way from an initial 256 essentially different solutions, as will be explained later when the list itself is being discussed.

(*k*) *Quadsets formed by two pairs of mutually-complementary quads*

Quadsets which consist of two pairs of mutually-complementary quads, that is, those which belong to classification (*c*) above, are of particular importance. There are six and only six and they form three distinctive 'associate' pairs.† The quads of each quadset have *two* constructive common part sums as just defined, no associate pair of quadsets sharing a common part sum. In other words the two common part sums of the quadset and the two common part sums of its associate are all different and thus, taken together, are 9, 13, 15 and 16. It is convenient to establish these properties of quadsets formed by pairs of mutually-complementary quads here, independently of what follows.

† These are the quadsets Π1, Π2; Π3, Π4; Π5, Π6 shown in Appendix IV.

The six quadsets are all used as Frénicle quadsets, a property which holds only for quadsets of category (*c*), that is to say, for all other classifications there are quadsets which do not form Frénicle quadsets although some of these occur as row-quadsets or column-quadsets. It will be shown further, in the course of the argument developed in §3, that in all the 432 pandiagonal or semi-pandiagonal magic squares, four of the five quadsets formed by the rows, the columns, the principal and short broken diagonals, the Frénicle corner squares and the Frénicle extended corner squares respectively form two (necessarily different) associate pairs from the three associate pairs of quadsets referred to above.

We first find all possible compatible sets of four quads consisting of two pairs of mutually-complementary quads. The first pair contains four numbers $1 \leqslant t < u < v < w \leqslant 8$ together with their complements written as w', v', u', t'. To make two quads from these numbers either (i) $t+w = u+v$ ($= z$, say) so that t, w, v', u' and u, v, w', t' each add to 34, or (ii) $u+v+w = 17+t$ in which event the quads are u, v, w, t' and t, w', v', u'. (Since $v+w \leqslant 15$, the element added to 17 must be the smallest.)

Note that when $t+w = u+v$, then $t+u+v+w = 2z$ is even, whereas when $u+v+w = 17+t$ they sum to $17+2t$, an odd number $\geqslant 19$. The other pair of mutually-complementary quads involves similarly numbers between 1 and 8. Since $1+2+\ldots+8 = 36$, both pairs cannot be of type (ii), since the sum would then be at least 38; nor can one pair of quads be of type (i) and the other of type (ii), since then the sum would be odd. Thus both pairs must be of type (i). For the second pair denote by Z the number corresponding to z. Since $2z+2Z = 36$, $z+Z = 18$. We may choose $z \leqslant 9 \leqslant Z$. Moreover $z \geqslant 5$, for otherwise there are no two compatible pairs of numbers adding to z. The complete range of possibilities (with choices which give no compatibility in square brackets) is as follows:

$z = 5 = 1+4 = 2+3$, $Z = 13 = 5+8 = 6+7$, leading to the quadset

1 4 14 15	2 3 13 16	5 8 10 11	6 7 9 12
(5 15 16)	(5 15 16)	(13 15 16)	(13 15 16);

$z = 6 = [1+5] = [2+4]$, $Z = 12 = [4+8] = [5+7]$, giving no compatibility;

$z = 7 = 1+6 = 2+5 = [3+4]$, $Z = 11 = 3+8 = 4+7 = [5+6]$, leading to

1 6 12 15	2 5 11 16	3 8 10 13	4 7 9 14
(7 13 16)	(7 13 16)	(11 13 16)	(11 13 16);

$z = 8 = 1+7 = [2+6] = 3+5$, $Z = 10 = 2+8 = [3+7] = 4+6$, leading to

1 7 12 14	3 5 10 16	2 8 11 13	4 6 9 15
(8 13 15)	(8 13 15)	(10 13 15)	(10 13 15);

$z = 9 = Z = 1+8 = 2+7 = 3+6 = 4+5$, all of which are compatible and leading to three choices of quadsets, namely

1 8 10 15	2 7 9 16	3 6 12 13	4 5 11 14
(9 11 16)	(9 11 16)	(9 15 16)	(9 15 16);
1 8 11 14	3 6 9 16	2 7 12 13	4 5 10 15
(9 12 15)	(9 12 15)	(9 14 15)	(9 14 15);
1 8 12 13	4 5 9 16	2 7 11 14	3 6 10 15
(9 13 14)	(9 13 14)	(9 13 16)	(9 13 16).

These six are thus the only compatible quadsets consisting of two pairs of mutually-complementary quads. The part sums are shown beneath the quads. For each quadset there are two common part sums (which must therefore, as has just been shown, be two from 9, 13, 15, 16). The analytical proof of this involves three alternatives and is not worth going through. The quadsets form 'associate pairs' having the important property that each element of a quad in one quadset of an associate pair appears in one and only one quad of the associate quadset. The associate pairs can be identified by their quads containing the element 1; namely the two quadsets containing the quads 1 8 11 14 and 1 6 12 15 respectively; the two containing the quads 1 8 10 15 and 1 7 12 14 respectively; and the two containing the quads 1 8 12 13 and 1 4 14 15 respectively. They appear in the list of all Frénicle quadsets labelled Π_1 to Π_6 in the order stated above.

We notice further from the arrangements on the array that each of the quads within the quadsets and so each of the quadsets as a whole is reversible, that is to say, as has been explained above, if the quadsets are turned through 90° on the array, they transform either into themselves – remaining unchanged Π_1, Π_2 – or into their associate pairs – Π_3 into Π_4, Π_5 into Π_6 and *vice versa*. In analogy with the transforms U, S, US on *solutions* described earlier, if we perform these transforms on the normal array itself to give the four different arrays

				U				S				US			
1	2	3	4	1	3	2	4	6	5	8	7	6	8	5	7
5	6	7	8	9	11	10	12	2	1	4	3	14	16	13	15
9	10	11	12	5	7	6	8	14	13	16	15	2	4	1	3
13	14	15	16	13	15	14	16	10	9	12	11	10	12	9	11

then, again, each of the quads of group 2 and so each of the Π-quadsets formed from them remains invariant within the set of six Π-quadsets, S leaving the quadsets unchanged, U and US interchanging the two pairs of associate quadsets Π_1 with Π_2 and Π_3 with Π_4, while leaving the associate quadsets Π_5, Π_6 unchanged.

3. The multiplicities of Frénicle quadsets by genus

(*a*) *Constructing solutions – the cross*

The four Frénicle quads have a vital characteristic additional to those of the other defined quads within a magic square, namely that each row, column and principal diagonal is made up of two pairs of numbers, each of which comes from a different Frénicle quad. Thus in the top row of a magic square one pair of the numbers belongs to the corner quad; the other pair to the vertical quad. In the bottom row, each pair of numbers is the supplement of the corresponding pair in the top row. Thus the corner and the vertical Frénicle quads must have at least one part sum in common, namely the Frénicle equality $a+d = o+p$ or its complement $n+q = b+c$ as described in §2(*b*).

On applying the principle of the equality of part sums throughout, we readily arrive at the following rules:

(i) the corner quad has each of its three part sums equal to and linked with a part sum of each of the other three Frénicle quads;

(ii) the same property holds for the centre quad;

(iii) the vertical and horizontal quads each have a part sum in common and linked with the corner and centre quads, but their other part sums need not be repeated.

It follows therefore that

(iv) a necessary (although not sufficient) condition for a quadset to form a *Frénicle quadset* is that, of the four quads of the quadset, two and only two can have a part sum which is not repeated elsewhere in the set.

We can thus describe a group of magic squares by its set of four Frénicle quads (quadset). We note that a Frénicle quadset is invariant with respect to the transformations U (which leaves each Frénicle quad unchanged) and S (which interchanges the roles of the Frénicle quads but leaves the Frénicle quadset as a whole unchanged). Moreover, if the Frénicle quadset is self complementary, it is invariant with respect to the replacement of the magic square by its complement so that this replacement produces no new square. If the quadset is not self complementary, replacing it by its complement produces an essentially different square.

Given any Frénicle quadset obeying the three rules for part sums stated above, a number (to be called the multiplicity) of essentially different magic squares may be constructed from it. To display the method of construction we arrange the part sums which are used in the form of a cross as illustrated. In the centre we place the part sum used in the principal diagonal i.e.

	corner and vertical	
corner and horizontal	corner and centre	centre and vertical
	centre and horizontal	

that shared by the corner and centre quads; at the top the part sum shared by the corner and vertical quads; on the left that used in the outer columns shared by corner and horizontal quads, on the right that used in the inner columns shared by centre and vertical quads and at the bottom that for the inner rows shared by centre and horizontal quads. Thus the corner quad has the three part sums appearing toward the left top (including the centre) of the cross, the centre quad has the three part sums toward the bottom right of the cross, the vertical quad has the top and right as part sums and the horizontal quad the left and bottom as part sums.

To construct a magic square from the cross, it is perhaps easiest to start with the corner quad, defined, as has been said, by the Frénicle quad supplying the three part sums appearing in the top left of the cross. The four numbers of this corner quad should be arranged so that the pairs of diagonally opposite numbers which we shall call the 'diagonal' part sum fit the part sum placed at the centre of the cross, and the top (and bottom) pairs fit the part sum placed at the top of the cross. Because of the complementarity this arrangement is not unique, but this does not matter as the transformations of rotation and reflection take care of this. Next, put in the centre quad to fit the diagonals and also take care to allow for any part sum which may be shared with the vertical quad or with the horizontal quad, which happens if two opposite arms of the cross are identical. There is still an ambiguity, but the choice can be made arbitrarily, since the transformation U takes care of this. The vertical and horizontal quads are now readily filled, completing the magic square.

As an example, consider the particular set of four quads given below which, as we can verify, satisfy the necessary conditions given in §2 to form a Frénicle quadset leading to a solution. The quads are written in such a manner that they can be respectively corner, centre, vertical and horizontal quads in that order. The part sums are shown below their quads.

3 8 10 13	2 7 9 16	4 5 11 14	1 6 12 15
(11 13 16)	(9 11 16)	(9 15 16)	(7 13 16)

To construct the cross, note that only the first and second quads can in this example be the corner or centre quads, these being the only two having no unrepeated part sums. Which of these two we choose for the corner quad and which for the centre quad is irrelevant as the transformation S takes care of this. The only choice for the diagonal part sum is thus 11, since otherwise we could not deal with the part sums 16 of the vertical and horizontal quads. Thus the only relevant cross is

	16	
13	11	9
	16	

where the top and the bottom of the cross are identical because the part sum linking the quad chosen as the corner quad to that chosen as the vertical quad is the same, namely 16, as the part sum which links what must then be the centre quad to the horizontal quad – the part sum 16 in this instance being common to each of the four quads.

The stages in the construction of the magic square are thus

3	•	•	13	3	•	•	13	3	14	4	13
•	•	•	•	•	7	9	•	6	7	9	12
•	•	•	•	•	2	16	•	15	2	16	1
10	•	•	8	10	•	•	8	10	11	5	8

The part sums are thus crucial, the cement which holds a magic square together. Unhappily, the analysis cannot be based solely on them. For, firstly, to any part sum triad not containing 17 there correspond two (mutually-complementary) quads rather than one; and, secondly, more seriously, the compatibility of any quads resulting from a set of four part sum triads can only be established by constructing the quadset which is therefore preferable as the primary building block.

(*b*) *The Frénicle-quadset genera*

In §2(*d*) we defined six classifications for all possible quadsets. Consider now only Frénicle quadsets. They, along with any other four compatible quads each of which has numbers adding to 34, must conform to one of these six classifications. The first, namely (*a*) where the quadset consists of four self-complementary quads, yields quadsets which, as will be shown later, need to be divided into two distinct 'genera' for Frénicle quadsets which we name Θ and Φ respectively. Conditions (*b*) and (*c*) define genera which we call X, Π respectively. The other two conditions (*d*) and (*e*) give rise to Frénicle quadsets which we have named respectively Λ and Ω; Λ because this proves to be an empty category (the 'lost' genus), while Ω is the 'last' genus. In all, 864 solutions belong to the four genera Θ, Φ, X, Π. To these 864, genus Ω adds a further 16 thus accounting for all Frénicle's 880 solutions.

We now consider, by using the cross, the construction of Frénicle quadsets of genera Π, Θ, Φ, X in that order, examining their multiplicities and their 'patterns' defined by links between complementary numbers within solutions, and establishing that all Frénicle quadsets in these genera have quads with a common part sum as do all rows and all columns respectively. Similar treatment for genera Λ, Ω follows, but the proof that the quads of these latter quadsets have a common part sum (which is the basis of our method of determining the population sizes) requires additional considerations to those which suffice for the first four genera. Once the existence of a common part sum for all Frénicle quadsets is established, their enumeration and identification for each genus becomes a straightforward process of using the principles of the 'constructive pairs' of part sums 9, 13, 15, 16, 17 (§2(*g*)) in a manner which will be described. This then gives the number of solutions belonging to each genus as required.

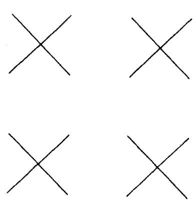
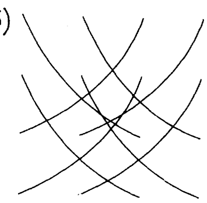
(*c*) *Self-complementary quadsets*

Genus Π. We define this genus as consisting of two pairs of mutually-complementary Frénicle quads. Since the pairs of mutually-complementary quads necessarily have identical part sums there can be no unrepeated part sums in genus Π quadsets. However, since both the corner quad and the centre quad must be linked to all other quads in the quadset, the two pairs of mutually-complementary quads must have *two* part sums in common. The part sum 17 does not occur since no quad is self complementary. Moreover one part sum of each pair must be individual to that pair, since no two quads can have all part sums identical unless they are mutually complementary. Thus the part sum structure can be represented as $(\alpha\,\beta\,\gamma)$ $(\alpha\,\beta\,\delta)$ $(\alpha\,\beta\,\gamma)$ $(\alpha\,\beta\,\delta)$, where each letter corresponds to a different number less than 17. No quad is here specially qualified to contain the diagonal part sum, and the two quads which form a pair are interchangeable, having the same part sums. The possible crosses are thus

$$\begin{array}{ccc} & \delta & \\ \gamma & \alpha & \gamma \\ & \delta & \end{array} \qquad \begin{array}{ccc} & \delta & \\ \gamma & \beta & \gamma \\ & \delta & \end{array} \qquad \begin{array}{ccc} & \delta & \\ \alpha & \gamma & \beta \\ & \delta & \end{array} \qquad \begin{array}{ccc} & \gamma & \\ \alpha & \delta & \beta \\ & \gamma & \end{array}$$

The first two, evidently different from the second two, have one pair of mutually-complementary Frénicle quads as corner and central quads respectively (and therefore their interchange is already taken care of by S), whereas the interchange of the vertical and horizontal quads is new. Thus, each of these has a multiplicity of 8. As regards the second pair of crosses, swapping either mutually-complementary pair of Frénicle quads is novel, resulting in a multiplicity of 16 for each of the pair. *The total multiplicity for genus* Π *quadsets is thus* 48.

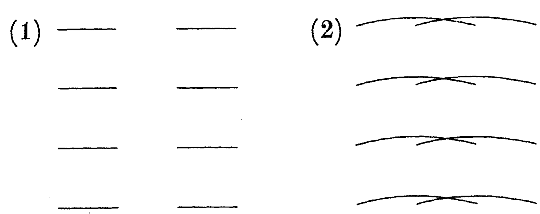
Consider now the patterns derived by linking each element of the magic square with Frénicle quadsets of genus Π with its complement.† The solutions derived from the first pair of crosses can be represented diagrammatically by the patterns

(4)

(5)


† The patterns which arise, 12 in all and well known, are shown in Appendix II.

where the lines represent links between complementary numbers and the designations (4), (5) respectively are those used by Andrews in 1908. We see that the transformation U acting on a solution with pattern (4) results in a solution with pattern (5) and vice versa, and that the transformation S leaves the patterns unchanged. It follows that each quadset derived from the first pair of crosses produces 8 solutions with pattern (4), sometimes called '*diagonal*' solutions which are semi-pandiagonal, and 8 solutions with patterns (5) which are *pandiagonal* since the four 'long broken' diagonals as well as the two short broken diagonals add to 34. For solutions with patterns (4), (5) not only the Frénicle quadsets, but also the row-quadsets and the column-quadsets, are formed by two pairs of mutually-complementary quads and must therefore form a pair of quadsets from the three associate pairs of genus Π defined (§2(*i*)) each having two common part sums. Moreover the Frénicle extended corner squares in solutions of pattern (4) and the Frénicle corner squares in solutions of pattern (5) form respectively with the Frénicle quads a second associate pair of quadsets of genus Π and thus also each have two common part sums. The Frénicle corner squares in solutions of pattern (4) and extended corner squares in solutions of pattern (5) together with the quadset formed by the principal and broken diagonals form two quadsets in which all quads are self complementary and thus have the common part sum 17.

For the second pair of crosses, where the corner quad and the vertical (or horizontal) quads are mutually complementary, the solutions can be represented diagrammatically by the patterns



the designations (1), (2) being those used by Andrews. Here, again the transformation U interchanges the patterns, whereas the transformation S leaves them unchanged. For these patterns the row quadsets are self complementary and thus have the common part sum 17; the column quadsets and 'diagonals quadsets' are formed by two pairs of mutually-complementary quads and are thus of genus Π with two common part sums. Since the Frénicle corner and extended corner squares have elements which respectively add to 34, the solutions are semi-pandiagonal (§2(*b*)). Considerations similar to those for patterns (4), and (5) show that these patterns also give rise to two associate pairs of quadsets of genus Π and two quadsets which are self complementary, all thus having respectively either two constructive common part sums or a common part sum 17.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Genera Θ, Φ

Next consider Frénicle quadsets composed of four self-complementary quads so that the part sum 17 occurs in each quad. If the diagonal part sum can be 17, then the cross must be

$$\begin{array}{ccc} & \alpha & \\ \beta & 17 & \gamma \\ & \delta & \end{array}$$

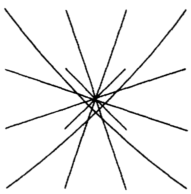
so that the part sums can be represented as ($\alpha\,\beta$ 17) ($\gamma\,\delta$ 17) ($\alpha\,\gamma$ 17) ($\beta\,\delta$ 17), where there is no unrepeated part sum, and the common part sum 17 in the third and fourth ('vertical' and 'horizontal') quads is not used in the construction process. If the 'diagonal' part sum is taken as α, so that the cross is

$$\begin{matrix} & 17 & \\ \beta & \alpha & \gamma \\ & 17 & \end{matrix}$$

leading to the part sums ($\alpha\,\beta$ 17) ($\alpha\,\gamma$ 17) ($\rho\,\beta$ 17) ($\delta\,\gamma$ 17), where ρ and δ need not be equal, we have a different set of criteria. Thus we define two distinct genera with four self-complementary quads forming the quadset: genus Θ with no unrepeated part sums, and genus Φ, say, in which each of two quads has an unrepeated part sum.

Consider genus Θ. If 17 is the diagonal part sum then the corner and centre quads can be either the first pair, namely ($\alpha\,\beta$ 17), ($\gamma\,\delta$ 17), or the second pair, namely ($\alpha\,\gamma$ 17), ($\beta\,\delta$ 17). Moreover, when the corners themselves have been settled, the equality of the part sum 17 with its complement (also 17) implies that the centre quad can be orientated in two alternative ways. With the transformations U, S, we can thus arrive at *a contribution of* 16 *to the multiplicity for genus* Θ *quadsets from the cross with its middle number* 17.

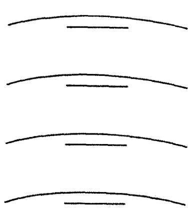
The solutions can then be represented by the pattern

(6)  which can be written as

$$\begin{matrix} a & b & c & d \\ e & f & g & h \\ h' & g' & f' & e' \\ d' & c' & b' & a' \end{matrix}$$

where (6) is Andrews's designation and $a' = 17-a$, etc. These are the *symmetrical* squares and are also semi-pandiagonal. The row quadsets and column quadsets, and the Frénicle corner and extended corner quadsets form two associate pairs of quadsets of genus Π and thus each have two common part sums. The 'diagonals quadsets' are self complementary and thus have a common part sum 17.

If a part sum different from 17 is chosen from the diagonal part sum there are then four options α, β, γ, δ, but for each of these the contributing Frénicle quads are determined. However, the occurrence of 17 as a part sum within the vertical and the horizontal quads as shown by the second of the crosses above means that each of these four arrangements leads to a multiplicity of 4×2 (from U) $\times 2$ (from S) $= 16$. The solution can be represented diagrammatically by the pattern

(3)  which can be written as

$$\begin{matrix} a & b & b' & a' \\ e & f & f' & e' \\ i & k & k' & i' \\ n & o & o' & n' \end{matrix}$$

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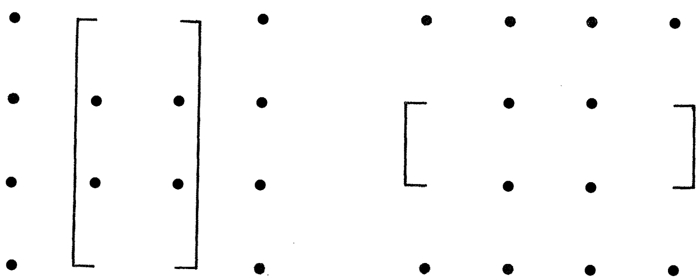
where Andrews's designation (3) is used and $a+f = k+n$. Of these $4\times 16 = 64$ solutions 2×16 will be semi-pandiagonal with $b+i = i+o$ and $2\times 16 = 32$ will not be so. For, if we interchange the two middle numbers of the top and the bottom rows, or interchange the two middle numbers within the outer columns thus:



while a semi-pandiagonal with pattern (3) remains magic and still retains pattern (3), its semi-pandiagonal property is destroyed. Making the two pairs of interchanges in succession restores this property, the Frénicle quads having remained unchanged throughout. *The total multiplicity for genus Θ has now emerged as* $16+4\times 16 = 80$.

Solutions of pattern (3) have rows which are all self-complementary quads and thus have a common part sum 17, and column quadsets formed of two pairs of mutually-complementary quads which thus have two common part sums. The 'diagonals quadsets' of the semi-pandiagonal solutions with pattern (3) are also formed of two pairs of mutually-complementary quads and thus form with the column quadsets associate quadsets of genus Π. The Frénicle corner and extended corner squares then also form associate quadsets of genus Π. The quadsets in genus Θ and so also the related solutions are all reversible.

Consider now genus Φ, namely quadsets with four self-complementary quads and two unrepeated part sums. The diagonal part sum must be α as shown in the second of the two crosses shown above, since the vertical and horizontal quads are the only permissible homes for the unrepeated part sums. Again, the occurrence of the part sum 17 *produces the multiplicity* 16 and the resulting solutions have the pattern (3). Since, here, the vertical and horizontal quads do not have a common part sum other than 17, they cannot be semi-pandiagonal. None of the Frénicle quadsets of genus Φ are reversible and so none of the solutions are reversible; the rows have a common part sum 17 and the columns, being two pairs of mutually-complementary quads, have been shown to have two common part sums. The diagonals do not form part of a quadset.

Genus X

Next consider Frénicle quadsets composed of two self-complementary quads and one pair of mutually-complementary quads. The first two share the part sum 17, the other pair has identical part sums all less than 17. This second pair cannot contain the diagonal part sum, for otherwise their other two (identical) part sums would have to link them to the part sums different from 17 in the first pair, giving both of them identical part sums which is not allowed. If the two self-complementary quads both contain the diagonal part sum then this diagonal part sum must be 17 and the part sum model must be ($\alpha\ \beta$ 17) ($\alpha\ \gamma$ 17) ($\alpha\ \beta\ \gamma$) ($\alpha\ \beta\ \gamma$) giving the cross

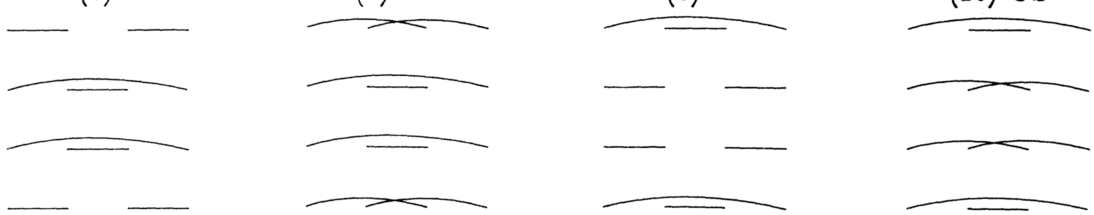
$$\begin{matrix} & \alpha & \\ \beta & 17 & \gamma \\ & \alpha & \end{matrix}$$

Call any part sum which occurs three times only a *triplet*. We shall find in §4 (b) that no solution with a Frénicle quadset of genus X can exist for which the quads have a common part sum (here α) *and* in which there are *two* triplets (here β, γ) and this model therefore leads to a nil set. The alternative possibility is that the diagonal part sum arises from one of the self-complementary quads and one of the pair of mutually-complementary quads. The part sums can then be represented by ($\alpha\ \beta\ \gamma$) ($\alpha\ \beta$ 17) ($\alpha\ \beta\ \gamma$) ($\omega\ \beta$ 17), giving the cross

$$\begin{matrix} & \beta & \\ 17 & \alpha & \gamma \\ & \beta & \end{matrix}$$

We note that the part sum β occurs four times and is thus a common part sum, while α occurs three times forming a *triplet*, one of the αs, together with ω, being unused in the cross. For the multiplicity we arrive at a factor 2 each from the occurrence of the part sum 17, from the interchange of the two mutually-complementary quads (which, since one is on the diagonal and the other is not, is not duplicated by S), from U, and from S, giving a total multiplicity 16. The links between equal part sums show that all the rows must be self complementary. Since the corner quad *or* centre quad is self complementary we have thus only four choices of patterns namely,

(7) (9) U (8) S (10) US



These are the patterns numbered as shown by Andrews. They form a set of four obtained from one another by the transforms U, S, US. Inspection makes clear that not only do the Frénicle quadsets have the common part sum β of the part-sum model (in the Frénicle general square notation we have here $a+n = f+k$, etc.) and the rows being self complementary have the common part sum 17, but the columns as indicated in the first two diagrams also have the *same* common part sum as the Frénicle quadsets, having however no complementarity. The quadsets are reversible when and only when the elements in the principal diagonals are reversible (and then form the solutions shown in Category Two of the solution list).

The multiplicities for the solutions whose Frénicle quadsets are of genera Π, Θ, Φ, X are thus 48, 80, 16, 16 respectively. We shall find that they give 864 solutions in all, 48 solutions of each pattern (4), (5), (6); 96 of each pattern (1), (2); 304 of pattern (3), 192 of these latter with Frénicle quadsets of genus Θ and 112 with Frénicle quadsets of genus Φ; and 56 of each of the patterns (7), (8), (9), (10). *All* solutions with these patterns are thus accounted for, but this is not proved until all other possible additional solutions are shown to have patterns different from these ten.

These 864 solutions with Frénicle quadsets as defined by the four genera we call *regular* (in contrast to the remaining solutions which are known in the literature as *irregular*). As has been shown, the regular solutions have patterns which are symmetrical about both the vertical and the horizontal central axes of the solutions. More importantly for our purpose, it follows from their definition that their Frénicle quadsets (and their rows and columns respectively) are composed of quads having a common part sum. It is this property of their *Frénicle quadsets* which makes their identification (and hence the enumeration and construction of the 864 solutions) a simple matter, as will be shown in §4.

(*d*) *Non-self-complementary quadsets*

We have still to consider possible solutions whose Frénicle quadsets belong to the two remaining classifications, namely (*d*) having one self-complementary quad, the other three having no complementarity, which we have called genus Λ; and (*e*) having four quads with no complementarity anywhere, which we have called genus Ω. For these genera it is not self-evident from the part-sum models for possible Frénicle quadsets (as it was for those of the first four genera just considered) that the quads must have a common part sum, a property which, as has already been said, makes enumeration and identification simple. The rest of this section is devoted to establishing this property for the two remaining genera.

Our proofs are based on the complementarity links between elements within and between quads. It is helpful to list a number of lemmas, some so obvious that no proof is given.

(i) If there is at least one link within a quad, then there are two and it is self complementary. This clearly applies to any 'line' within a solution, that is, to any row, column or principal diagonal whose elements must add to 34, as well as to Frénicle quads.

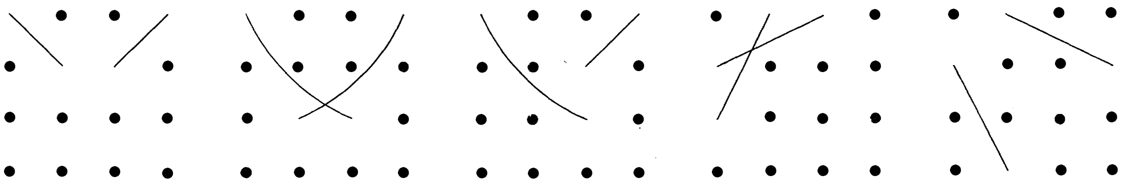
(ii) If there are at least three links between two quads, then there are four links and the quads are mutually complementary.

(iii) If there are two links between two Frénicle quads whose elements together form two lines then they cannot connect two elements of one quad lying along the other line. For example, on using the established Frénicle notation

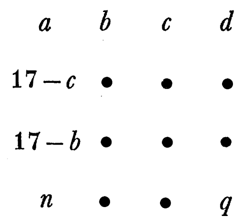
$$\begin{matrix} a & b & c & d \\ e & f & g & h \\ i & k & l & m \\ n & o & p & q \end{matrix}$$

for the elements of a solution, since $a+q = g+k$, the elements a, g and k, q cannot be complementary links adding to 17.

(iv) If two non-self-complementary quads are not mutually complementary, a further set of double links as illustrated below together with their reflections, rotations and transformations by U, S are also precluded,



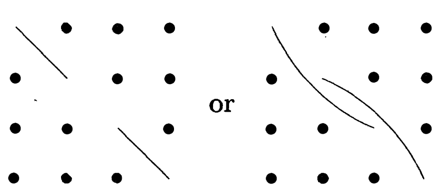
the first three because completing the links along the diagonals makes the two (corner and centre) quads mutually complementary (the second is the U transform of the first), and the last two which are S transforms of one another because



gives $a+n = c+b = 34-a-d = n+q$, that is, $a = q$ which is precluded.

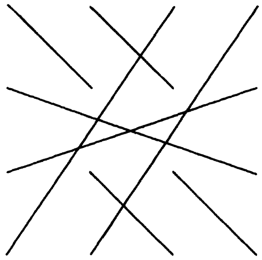
Genus Λ

Consider now possible Frénicle quadsets of genus Λ. Take any one of the three non-self-complementary quads. Its four elements require four complements with the quadset none of which can lie in the quad itself (for otherwise it would be self complementary) nor in the self-complementary quad which contains all its own complements; nor can more than two lie in either of the two other quads for then there would be a mutually-complementary pair. Thus each of these three quads must be 'double linked' to the other two, and it follows that the three then share a part sum. The part sum model can then be written either as $(\alpha\, \beta\, \gamma)$ $(\alpha\, \delta\, \epsilon)$ $(\alpha\, \beta\, \delta)$ $(\gamma\, \epsilon\, 17)$ where the three non-self-complementary quads must necessarily have a part sum in common but this is *not* shared with the self-complementary quad; *or* as $(\alpha\, \beta\, \gamma)$ $(\alpha\, \beta\, \delta)$ $(\alpha\, \delta\, \beta)$ $(\alpha\, \gamma\, 17)$, where there is a part sum in common. We are concerned here to eliminate the first possibility. The self-complementary quad has 17 as a part sum unrepeated elsewhere and so cannot be a corner or central quad. Without loss of generality, we may take it to be the horizontal quad. The exclusions illustrated in (iii) and (iv) above show that the double link between the corner and central quads (which must be two quads from the three quads with the shared part sum and thus must share a part sum) cannot involve two adjacent corners. The only possibilities which we seek to exclude must thus have links

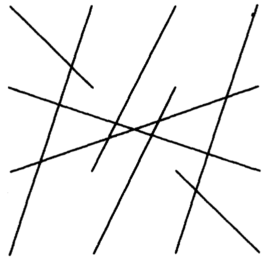


which are the U transforms of one another and do not have to be dealt with separately. On following the rules of exclusion shown earlier, the only possibilities for the completion of the double links are thus

Λ(*a*)



Λ(*b*)



For neither Λ(*a*) nor Λ(*b*) do the quads of the Frénicle quadsets have a common part sum. However the rows of Λ(*a*) have a common part sum, as have the columns of Λ(*b*), for if one row contains the complements of two elements of another row, then the sum of those two elements is the same as the sum of the remaining two elements of the first row mentioned. Hence, from the arguments of §2 (viii), each row of Λ(*a*) and each column of Λ(*b*) must consist of two even and two odd numbers. On remembering that linked (complementary) elements have opposite parity and that the sum 34 requires 0, 2 or 4 elements of each parity, the potentially possible patterns Λ(*a*), Λ(*b*) are readily seen not to be possible. For, write 1, 2 to denote numbers of opposite parity, then the top row of Λ(*a*) can take one of the three forms 1 1 2 2, 1 2 2 1, 1 2 1 2. On using only the direct links, we then have

1	1	2	2		1	2	2	1		1	2	1	2
•	2	2	•		•	2	1	•		•	2	1	•
•	•	•	•		•	•	•	•		•	•	•	•
1	1	•	•		1	2	•	•		2	1	•	•

The first of these fails because the outer elements of the second row must have the same parity, which means that the inner elements of the first column cannot have the same parity, and the opposite argument rules out the third alternative. In the second alternative it is impossible to choose the parity of the second element of the third row to suit both the column and the diagonal in which it lies.

For Λ(*b*), the skeleton squares can be taken as having even/odd elements as shown:

1	1	•	•		1	2	•	•		1	1	•	•
1	2	•	1		2	2	•	1		2	•	•	2
2	•	•	2		2	•	•	1		1	•	•	1
2	•	•	•		1	•	•	•		2	•	•	•

where the *columns* must all have two odd and two even elements, and the rows and the Frénicle quads must all add to an even number. It is quickly seen that these rules cannot be satisfied.

This has thus established that if a solution has a Frénicle quadset of genus Λ, then the quads forming the quadset have a common part sum.

We shall find in the next section that there are no solutions which can satisfy these conditions. We need not therefore concern ourselves with hypothetical possible patterns or with their multiplicities.

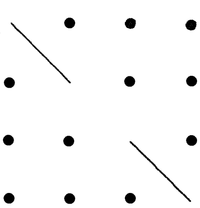
Genus Ω

Finally, consider possible Frénicle quadsets of genus Ω, in which the quads have no complementarity anywhere. The four complements of the elements of any one quad may be distributed over the three others either as (2, 2, 0) or as (2, 1, 1). These linkages can be combined easily into two arrangements: either each quad is double linked to two of the others (ensuring a common part sum for all quads); or there are two pairs of double-linked quads, all other links being single and then there need be no common part sum. It is the possibility of there being no common part sum which we have to examine and dismiss.

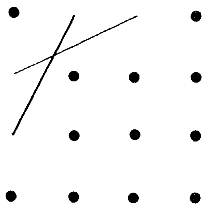
The most general part sum model for Frénicle quadsets of genus Ω is $(\alpha\,\beta\,\gamma)$ $(\alpha\,\epsilon\,\eta)$ $(\beta\,\epsilon\,\rho)$ $(\gamma\,\eta\,\delta)$ where no two triads are the same (thus ensuring that no two quads are mutually complementary), and no part sum equals 17 (thus ensuring that there is no self-complementary quad). The triads are to be regarded strictly in order as the part sums of the corner, centre, vertical and horizontal quads respectively. The interchanges (i) β with γ, ϵ with η, ρ with δ swap the last two quads, that is the vertical and horizontal quads; (ii) ρ with ϵ, γ with η swap the first pair (corner and central). Both (i) and (ii) leave the set of four triads unaltered.

Consider coincidences between part sums. If $\alpha = \rho = \delta$ or if $\beta = \eta$ or if $\gamma = \epsilon$, there is a common part sum (but no triplet) and, in our endeavour to exclude the possibility of there being no common part sum, these need not be considered further. Some other coincidences are forbidden since no triad may contain a repeated part sum. Permitted coincidences not leading to a common part sum are only $\rho(\neq \delta) = \alpha$ or γ or η; and $\delta(\neq \rho) = \alpha$ or β or ϵ, each of these conditions leading to a triplet. Although each singly is possible, only a few combinations can occur.

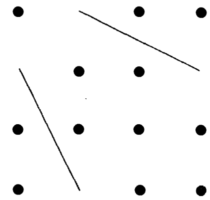
A common part sum can only fail to exist if there are merely two double links between Frénicle quads, since the alternative, four double links, ensures a common part sum. Suppose first that the two double links are between the corner and central quads and between the vertical and horizontal quads. Since the corner and central quads cannot have a part sum other than α in common, the double link must be on a line linking the two quads, that is, on a diagonal which can be chosen to be $a\,f\,l\,q$. On applying, if necessary, the transformation U, the links can be made $a+f = 17$ and $l+q = 17$ respectively, taken as



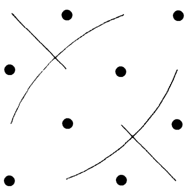
From Lemma (iv) the double links



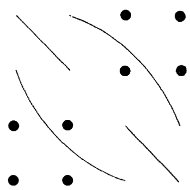
and their transforms by S, namely



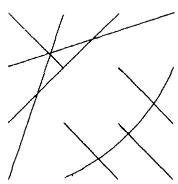
are excluded. Nor can the double link between the vertices and horizontal quads together with the double link above be



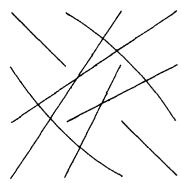
or



for then the only possibilities for the single links between the corner and vertical quad and between the corner and horizontal quad give



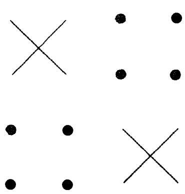
or



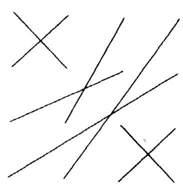
the first of which goes out at sight because of the incompatibility of the top row and first column, and the second of which goes out since, on using the equality of Frénicle opposing corner squares, these would be

$$\begin{array}{cccc} a & b & \bullet & \bullet \\ e & 17-a & \bullet & \bullet \\ \bullet & \bullet & 17-q & 17-b \\ \bullet & \bullet & 17-e & q \end{array}$$

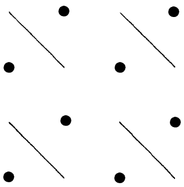
giving $2(b+e) = 34$ which is precluded. There remain only two other possibilities which go out equally simply:



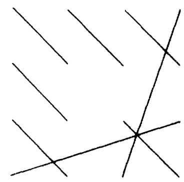
which can be completed only as



where the top row and the second column are incompatible, and

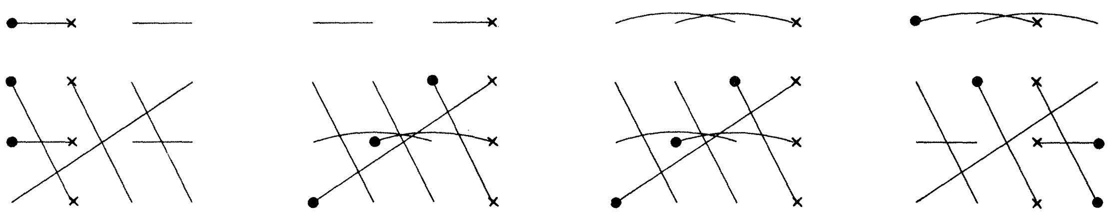


which can be completed only as



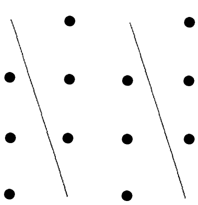
where the first and second columns are incompatible.

Alternatively the double links may be between the corner and vertical quads and between the central and horizontal quads. This alternative requires rather longer treatment. Suppose initially that the first pair shares only one part sum, namely β, and further that the second pair share only the one part sum η. Then the double links must make one outer and one inner row self complementary. On using, if necessary, a 180° rotation, we can make these the first and third rows. The four single complementary links between elements of the four quads then result in 'single shift' links between the now mutually-complementary second and fourth rows. If we omit the mirror images, the only potential patterns are then sufficiently

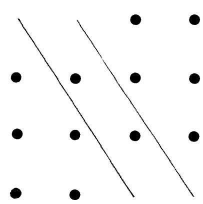


where here the symbols •, × indicate what are, as can be seen from the linkages, incompatible columns and/or diagonals, showing that all four patterns must fail.

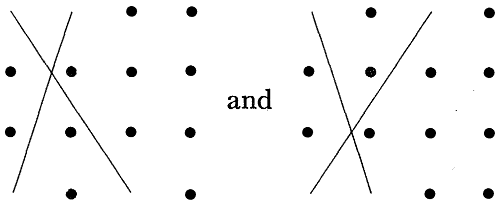
Next we consider what happens if the vertical quad shares a second part sum with the corner quad, the double links between the two creating the second shared part sum. As pointed out above, this means that either $\rho = \alpha$ or $\rho = \gamma$. Now ρ is the sum of two elements of the vertical quad neither in a row (which would give β) nor in a column (which would give ϵ), and thus are joined by links in a skew direction. In the outer quad, α is the sum of two elements in a diagonal. Thus the only double links which become possible with $\rho = \alpha$ must be



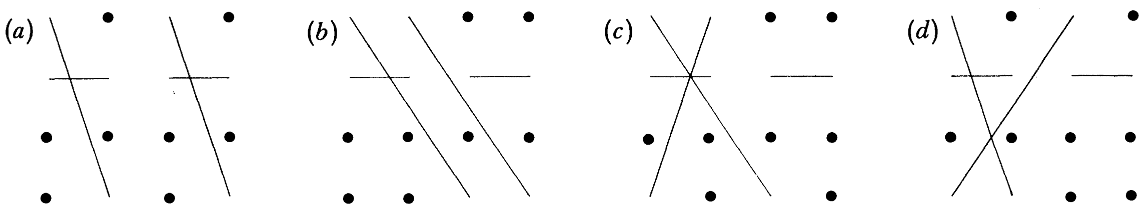
or its U transform



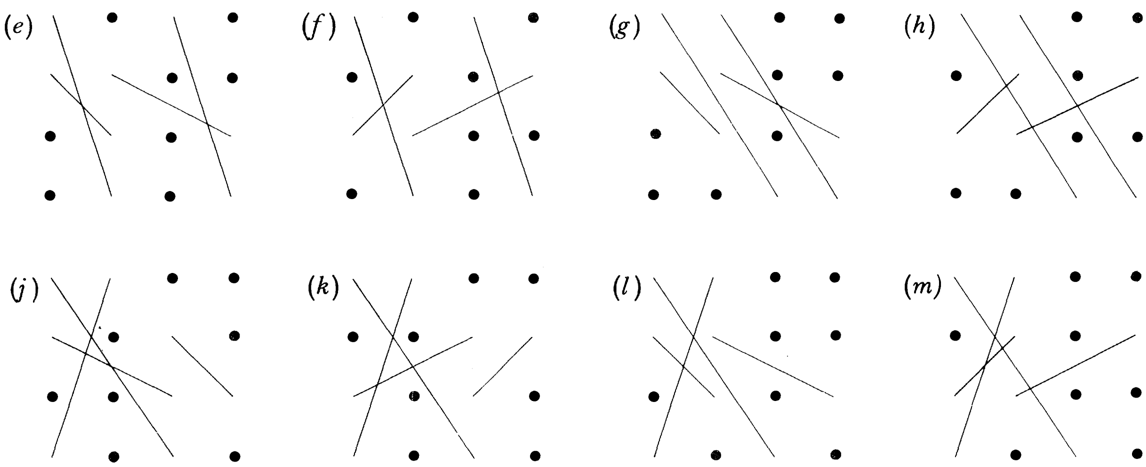
or their reflexions in the vertical through the centre of the square. Similarly the double link which becomes possible with $\rho = \gamma$ must involve two elements of the corner quad in a column since γ links the corner and horizontal quads. The links which then arise are thus



If there is no second shared part sum between the central and horizontal quads, their double links fill a row and we have just four essentially different possibilities (together with their reflections and U and S transforms which do not need to be treated separately), namely,



Finally we have to examine the possibilities when the corner-vertical and the centre-horizontal pairs each have a second shared part sum which are thus occupied by their respective double links. For the centre-horizontal pairs the possibilities are $\delta = \alpha$ or $\delta = \epsilon$, the corresponding diagrams being readily derived from those above by transformation S. We note however that, if $\rho = \alpha$ then $\delta = \alpha$ can be excluded as this gives a common part sum. We need therefore only to consider either $\rho = \alpha$, $\delta = \epsilon$; or $\rho = \gamma$, $\delta = \epsilon$, since $\rho = \gamma$, $\delta = \alpha$ is equivalent to the first of these by transformation S. We thus arrive at the set of potentially possible linkages:

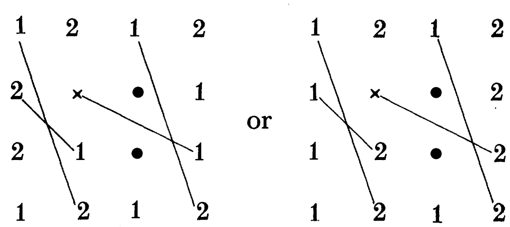


This exhausts all possibilities for solutions with Frénicle quadsets of genus Ω the quads of which would not have a common part sum. We thus have twelve essentially different skeleton patterns to demolish.

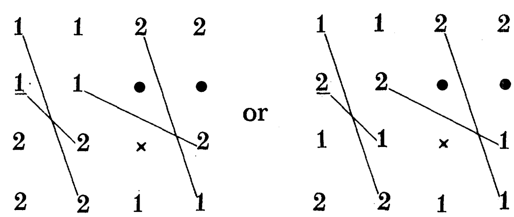
The last eight, (*e*)–(*m*) above, can be ruled out immediately by a simple process. We recall that since all lines in a square must add to 34, the elements in these lines must be all even, or all odd, or two odd and two even. We have shown in § 2 (*f*) that no *two* rows and no *two* columns can be all even or all odd. We have also shown that all Frénicle quads must have two odd and two even elements. As earlier, write 1, 2 to denote numbers of opposite parity, then the corners of any square can be written sufficiently in one of three ways, namely

(i)	1	•	•	1	(ii)	1	•	•	2	(iii)	1	•	•	2
	•	•	•	•		•	•	•	•		•	•	•	•
	•	•	•	•		•	•	•	•		•	•	•	•
	2	•	•	2		1	•	•	2		2	•	•	1

Consider then the possibilities for a square based on (*e*) above. With (i), the top and bottom rows would be all 1 and all 2 respectively, and the two middle rows must also be all 1 and all 2 respectively which is ruled out. With (ii), we arrive at

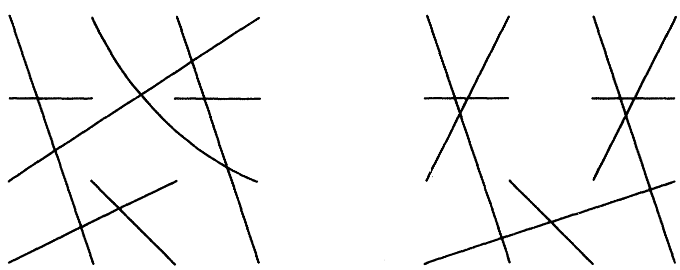


and the gaps cannot be filled without breaking the rules stated above. With (iii), we can start with



and there is failure at the position of the cross. All the other seven skeleton patterns lead similarly to total failure.

The skeleton patterns (*a*)–(*d*) require more detail. Consider (*a*). The links to comply with the conditions for genus Ω can be made in two ways, namely

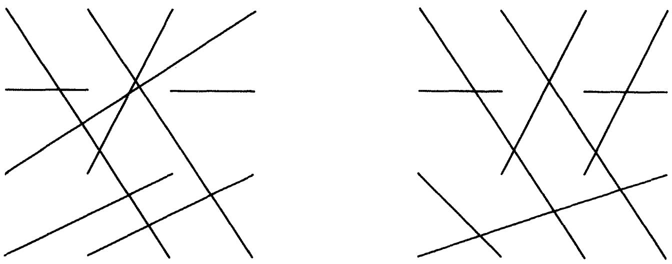


the second of which fails because the first two columns are incompatible. The first can be written as

a	b	c	d
e	$17-e$	$17-h$	h
$17-d$	$17-p$	$17-n$	$17-b$
n	$17-a$	p	$17-c$

From the bottom row, a diagonal, the last and the third columns, we have respectively, $a+c = n+p$, $d+n = h+p$, $d+h = b+c$ $c+p = h+n$; whence $d+p = b+n$, which with $d+n = h+p$ gives $2d = b+h$. Also $2p+n+h = a+c+d+n$, whence $2p = 34-b-h = 34-2d$, giving $p+d = 17$ which is precluded.

Now consider (b). The links can be made in two ways here also, namely



The first of these can be written

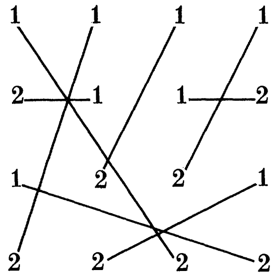
a	b	c	d
e	$17-e$	$17-h$	h
$17-d$	$17-c$	$17-n$	$17-o$
n	o	$17-a$	$17-b$

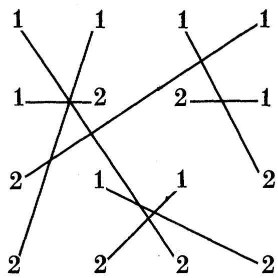
where Frénicle corner squares and the bottom row give $17+a+b = 68-b-n-o = 68-a-b-a-b$, giving $3(a+b) = 51$, whence $a+b = 17$ which is precluded. The second can be written

a	b	c	d
e	$17-e$	$17-h$	h
$17-o$	$17-c$	$17-d$	$17-n$
n	o	$17-a$	$17-b$

where the corners and the third column give $a+d+n = 17+b$ and $a+d+h = 17+c$, given $n-h = b-c = c-d$ by a diagonal, giving $2c = b+d$; and the first column and the corners give $a+e+n = 17+o$ and $a+d+n = 17+b$, giving $e-d = o-b$. But, from the second column, $b+o = e+c$, giving $2b = c+d$, whence $b = c$ which is precluded.

For (c), the possibilities from (i) are



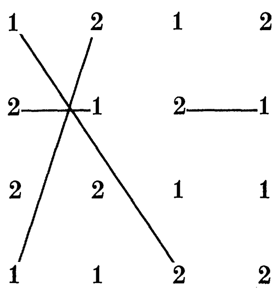


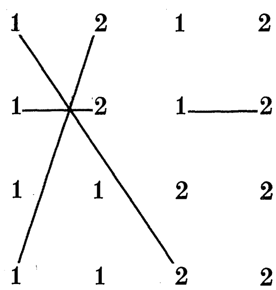
In the notation of the Frénicle magic square, namely

a	b	c	d
e	f	g	h
i	k	l	m
n	o	p	q

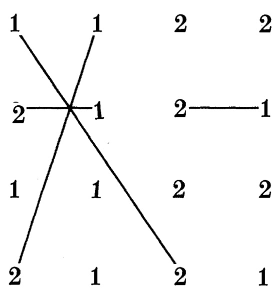
the first of these linkages gives $a+b = o+q = k+l$ and the Frénicle equalities give $k+l = e+h$, whence the rows have a common part sum. Similarly in the second, the links give $a+b = o+q = i+m$ and the Frénicle equalities give $i+m = f+g$, whence again the rows have a common part sum. But we have shown in Part Two that in all quadsets in which the quads have a common part sum the quads have two even and two odd elements. The top and the bottom rows of both the above linkages transgress this and thus rule out these possibilities.

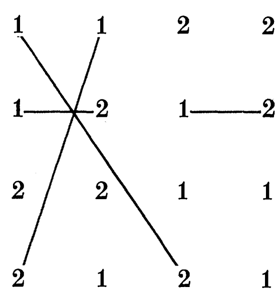
The possibilities for (ii) would be





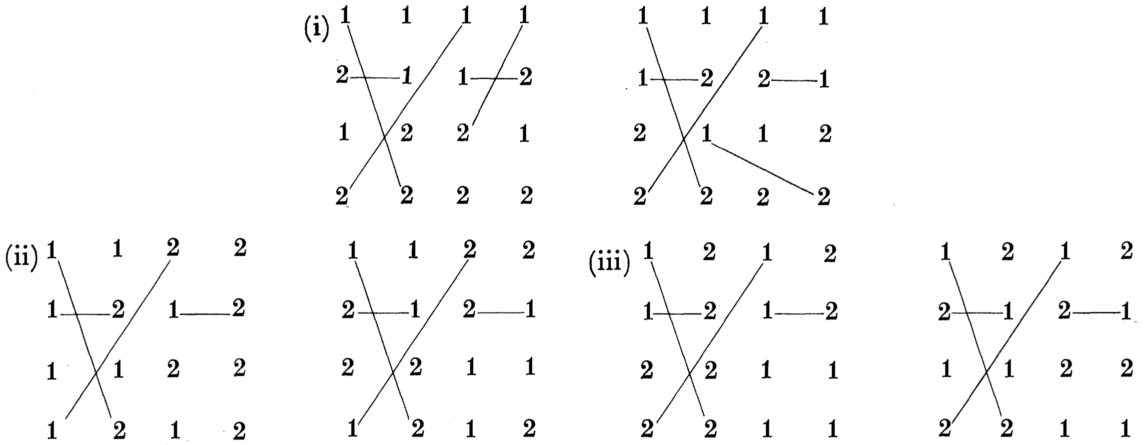
and for (iii)





All four of the above are instantly ruled out because the Frénicle diagonal equalities $a+q = g+k$, $d+n = f+l$ cannot be achieved with these parities.

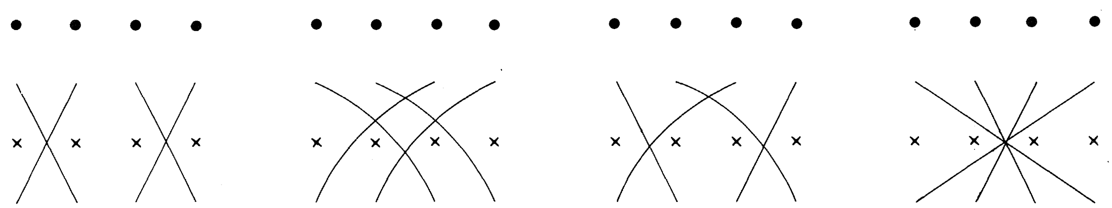
Finally, for (*d*), the possibilities are



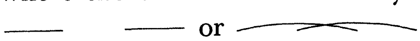
The link between the corner and centre quads in the first two squares can be only as shown and the remaining (fourth) centre elements have no vacant permissible link to the vertical quads. The last four possibilities are ruled out because the Frénicle diagonal equality $a+q = g+k$ as above cannot be satisfied with these parities.

This then concludes the proof that all Frénicle quadsets of genus Ω must have a common part sum. It establishes further that the only successful Frénicle quadset of genus Ω leads to the part sum triads $(\alpha\,\beta\,\gamma)$ $(\alpha\,\gamma\,\eta)$ $(\beta\,\gamma\,\rho)$ $(\gamma\,\eta\,\delta)$ with γ the common part sum, ρ, δ the unrepeated part sums and no triplet. Moreover, since there are no interchangeabilities other than those taken care of by the transformation U, S, US, the *multiplicities for Frénicle quadsets of genus* Ω *is four*.

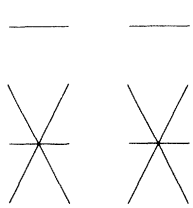
We still need to determine the potential patterns for solutions with Frénicle quadsets of genus Ω. They arise from the quadsets themselves which we now know must have quads with a common part sum and which cannot be such that the double links between two pairs of quads (which must exist) are between the corner and centre quads *and* between the vertical and horizontal quads. The double links must therefore be between the corner and vertical (or horizontal) quads and between the centre and horizontal (or vertical) quads. The possibilities together with their U transforms are therefore sufficiently represented by



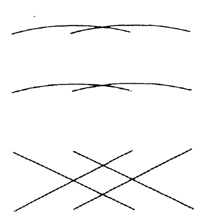
where the two rows marked by the symbols •, × are self complementary and have links



in some combination. The techniques used earlier in this section eliminate very simply all these possibilities other than



or its U transform



Here the rows have a common part sum, as do the columns since these latter form two pairs of mutually-complementary quads. These two surviving patterns for solutions are those named by Andrews (11), (12). They complete the set of all possible patterns for solutions, thus now proved to be necessary and sufficient without any recourse to discussion or knowledge of actual solutions. As far as we are aware, this is the first 'independent' analytic proof of the existence of twelve and only twelve patterns. As solutions with Frénicle quadsets of genus Λ will be shown not to exist, the work of this section has proved incidentally that

In all solutions the Frénicle quads, the rows and the columns have respectively at least one common part sum.

4. Enumeration of the Frénicle quadsets

(a) *Populations*

We have seen that any magic square must belong to one of the six genera and that each genus has a particular multiplicity so that the number of magic squares with Frénicle quadsets of a particular genus is its multiplicity multiplied by the number of essentially different Frénicle quadsets (the 'population') satisfying the characteristics of the genus. Frénicle quadsets of genus Π have been shown to have quads with two common part sums less than 17, while Frénicle quadsets of genera X, Λ, Ω have quads with one common part sum less than 17. For Frénicle quadsets of genera Θ, Φ the common part sum is 17.

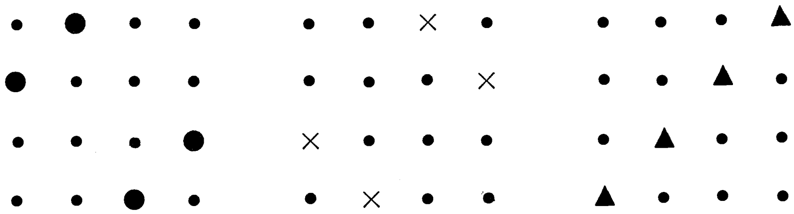
(b) *The populations for genera* Π, X, Λ, Ω – *the arrays*

Here the common part sum must be 9, 13, 15 or 16 (§2(*g*)) and each Frénicle quad must contain one out of four 'constructive' number pairs adding to the relevant part sum together with one out of the four number pairs which are their complements, each of the pairs occurring once and only once in each quadset. Call the constructive number pairs of the table in §2 the 'first number pairs' and the set of four number pairs which are their complements the 'second number pairs'. Then for each of the four relevant part sums we can construct a 4 × 4 *array of quads*, each row consisting of quads containing the same first number pair and each column consisting of quads containing the same second number pair. Each of these four arrays then contains all quads which have constructive part sums 9, 13, 15, 16 respectively. The arrays are shown on a pull-out page (Appendix III), the part sum being written beneath their quads, and supplemented by parallel arrays showing the regularities of the appearances of the part sums other than the relevant part sum common to all quads in the array concerned. The nature of the arrangement of the quads within each array as described above decrees that a compatible set of four quads must contain one and only one quad from each row of the relevant array and only one from each column.

Call the principal diagonal which runs from the top left-hand corner to the bottom right-hand corner of an array the *main diagonal*. We see that the quads in a main diagonal, being self-complementary, must have 17 as a part sum. Quads symmetrically placed about a main diagonal are mutually complementary and thus have identical part sums. Quads linked by a line parallel to the main diagonal also share a part sum, as do quads linked by a line parallel to the other principal diagonal, so that a lattice structure results.

Genus Π

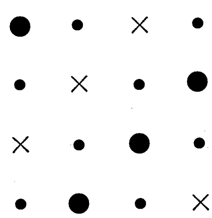
Each Frénicle quadset of genus Π has two common part sums (§2(*i*)) and must therefore appear in both of two arrays. In each array the quads must be symmetrically placed about the main diagonal in two (mutually-complementary) pairs. There are only three possibilities for this (since one quad of the quadset must lie in each row of the relevant array and one in each column), namely those which form the patterns indicated by the symbols ●, ×, ▲



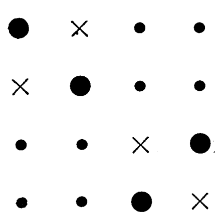
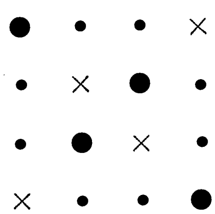
The connections being lattice lines show that the necessary equality of the part sums is assured. Thus each of the four common part sums 9, 13, 15, 16 can give rise to three quadsets of genus Π, resulting in $3 \times 4 = 12$ quadsets, but this has to be divided by 2, since each quadset appears in two different arrays corresponding to the two common part sums. This confirms that the population for Frénicle quadsets of genus Π is *six*. Since the multiplicity, as has been shown, is 48, the total number of magic squares arrived at is $6 \times 48 = 288$.

Genus X

Here there are two self-complementary Frénicle quads, necessarily lying on the main diagonal of relevant arrays, and a pair of mutually-complementary quads symmetrically placed about it. Furthermore, one of the self-complementary quads must share a part sum with the pair of mutually-complementary quads to give at least one triplet. If those three are on a lattice line, that is, forming the patterns

(i) 

a quadset of genus X necessarily results, the three like symbols on a lattice line giving the required triplet. Since each of the four arrays gives two quadsets with this pattern, there is a total of eight. The only other potentially successful patterns would be based on the patterns with two self-complementary quads and a pair of mutually-complementary quads as shown, namely

(ii)  (iii) 

where actual success is possible if, fortuitously as it were, there is coincidence between a part sum of the two mutually-complementary quads and one of the self-complementary quads. This never occurs in the 13- or 15-arrays, but three of the potentially successful patterns of (ii), (iii) above succeed in the 9-array and three succeed in the 16-array, leading thus to another six Frénicle quadsets of genus χ. *There is no possibility of four compatible quads giving two different triplets.* We thus have *fourteen* essentially different Frénicle quadsets of genus χ in all. With a multiplicity of 16 as shown earlier, this results in 224 magic squares with Frénicle quadsets of genus χ.

Genus Λ

Frénicle quadsets of genus Λ would have to have one and only one self-complementary quad (which must lie on the main diagonal of an array) and three quads with no complementarity. There are just eight essentially different available sets of positions for the quads within an array and having a common part sum (different from 17) which satisfies these conditions, namely those indicated by the four symbols ●, ×, ▲, ■ in the diagrams

●	×	■	▲		●	■	×	▲
▲	■	×	●		×	▲	●	■
×	●	▲	■		▲	×	■	●
■	▲	●	×		■	●	▲	×

In all eight of these quadsets one and only one pair of quads lie on the same lattice line and no two quads are symmetrical about the centres of the arrays. Thus, in all four arrays, all quadsets having the positions shown have more than two unrepeated part sums which is precluded for Frénicle quadsets. This establishes that there can be no solutions with Frénicle quadsets of genus Λ.

Genus Ω

Frénicle quadsets of genus Ω consist of four quads with no complementarity and there can be no triplet (§3(*d*)). The only possible positions for four compatible quads within an array (thus having a common part sum) and satisfying these conditions are

•	●	×	•		•	•	×	●
×	•	•	●	and	•	•	●	×
●	•	•	×		●	×	•	•
•	×	●	•		×	●	•	•

where the symbols ●, × represent different quadsets respectively. The first quadsets illustrated fail in the 9-array (triplet 15), and in the 13- and the 15-arrays with four unrepeated part sums, but succeed in the 16-array because of the 'fortuitous' repetitions of parallel lattice lines with shared part sums 11, 13 respectively, (giving the two essentially different complementary Frénicle quadsets Ω_3, Ω_4). Similarly the second two quadsets illustrated fail in the 13-, 15-, and 16-arrays, but succeed in the 9-arrays because of the repetitions of the part sums 14, 16 respectively in adjacent rows of the last column in this 9-array, (giving the two essentially different complementary Frénicle quadsets Ω_1, Ω_2). There are no other possibilities. We have thus arrived at the *four* essentially different Frénicle quadsets of genus Ω, and the explanation of the remarkable phenomenon (now that it has been proved that all Frénicle quadsets of this genus have quads with a common part sum) of the existence of only sixteen 'irregular' solutions, the multiplicity having been shown in (§3 (*d*)) to be four.

(*c*) *The populations for genera* Θ, Φ

Frénicle quadsets of these two genera have the common part sum 17, which, being its own complement, makes the methods used to enumerate the Frénicle quadsets for the other genera here less convenient. We have shown in (§3 (*b*)) that any specific Frénicle quadset of either genus leads to a set of magic squares of pattern (3) which can be written as

$$\begin{array}{cccc} a & b & b' & a' \\ e & f & f' & e' \\ i & k & k' & i' \\ n & o & o' & n' \end{array}$$

in the usual notation with $a' = 17-a$, etc., and where $a+f = k+n$ to ensure that the elements in the principal diagonals add to 34. The columns form two pairs of mutally-complementary quads and thus, as was shown in (§2 (*k*)) must be from among the six Π-quadsets already defined and must therefore have two common part sums. More particularly, any specific Frénicle quadset of genus Θ, but not of genus Φ, leads to a set of semi-pandiagonal magic squares of this pattern (§3 (*c*)) for which the condition $b+e = i+o$ must obtain and elements in the Frénicle corner squares and in the Frénicle extended corner squares add to 34 (§2 (*b*)). Hence, then $a+e = 34-b-f = k+o$ and $a+i = 34-b'-k' = b+k$ provide the two common part sums corresponding to each Π-quadset and a magic square with Frénicle quadset of genus Θ always exists. When the Frénicle quadset is of genus Φ and the square is not semi-pandiagonal, the equalities above do not hold and thus cannot form the two common part sums. Thus either $a+n$ or $b+o$ (or their complements) have to be involved, and one at least of the common part sums for a Φ must be different from those for a Θ derived from the same Π-quadset, if and when one or more Π-quadsets can be thus derived.

Call the columns of the square above A, B, B', A' and write the conditions for any (3) thus depicted as $n-a = f-k$, and the additional condition for a semi-pandiagonal (3) $o-b = e-i$. To enumerate the Frénicle quadsets of the two genera we look at each Π-quadset in turn and examine whether any of the differences between the elements of one quad (which can be taken as the column A) equal any of those of one of the two quads not complementary to A (which can be taken as column B). Any such equality specifies a Θ or a Φ. Since a Θ necessarily exists, the conditions for this, namely $n-a = f-k$, $e-i = o-b$, ensure that there is a pair of differences within the column A which are the same as a pair of differences within the column B. They use all elements in the two columns A, B and can be called the 'necessary' differences. Moreover the magic square

$$\begin{matrix} a & b & b' & a' \\ f & e & e' & f' \\ k' & i' & i & k \\ n' & o' & o & n \end{matrix}$$

is then also magic, essentially different from that above and with the same Frénicle Π-quadset, the columns being interchanged with the quadsets formed by the principal and short broken diagonals; the rows, the Frénicle corner squares and the Frénicle extended corner squares respectively being re-arranged internally. This will occur for any two Π-quadsets for which each quad of one quadset has one and only one element in common with each quad for the other quadset, namely if and only if the two Π-quadsets form one of the three 'associate pairs' already defined in Part Two, thus establishing that there are just three different Frénicle quadsets of genus Θ corresponding to these three associate pairs of Π-quadsets.

We can now proceed to examine each Π-quadset in turn, choosing arbitrarily for the columns A, B the quad which contains the element 1 and the quad which is not its complement with the smallest 'free' element, the transformations U, S looking after any other choices. Each quad then leads to six 'differences', one pair of differences in one group which is the same as a pair of differences in the other, which use the eight elements in the two quads once and once only and yield a Θ. These 'necessary matching differences' are shown starred in the working which follows. Any other matching differences, if and when they exist, are fortuitous and then yield Φs. They are marked with a dash. These can be of two kinds: those which involve one (but never both) of the differences used in creating the Θs, and those which do not. In the latter event the matching differences are entirely separate from those leading to a Θ, but they do not use all elements of the two quads and must therefore use one element in each quad twice.

Π_1: 1 8 11 14, 2 7 13 12 give differences 7 10* 13 3 6* 3, 5 11 10* 6* 5 1 respectively. The necessary differences 10*, 6* yield Θ_1 and there are no other matching differences to provide a Φ-quadset.

Π_2: 1 6 12 15, 3 8 10 13 give differences 5*′ 11 14 6 9 3*, 5* 7 10 2 5′ 3* respectively. The necessary differences 5*, 3* yield (as they must) the same Θ-quadset Θ_1 as above. The fortuitous matching differences given by the dashed 5s yield Φ_1 as listed.

Π_3: 1 8 10 15, 3 6 12 13 give differences 7′ 9* 14 2 7* 5, 3 9* 10 6 7*′ 1. The necessary differences 9*, 7* yield Θ_2, and the fortuitous matching differences given by the dashed 7s yield Φ_2 as listed.

Π_4: 1 7 12 14, 2 8 11 13 give differences 6* 11′ 13 5′ 7 2*, 6* 9 11′ 3 5′ 2*. The necessary differences 6*, 2* yield Θ_2 again as they must, and the *two* sets of fortuitous matching differences given by the dashed 11s and 5s yield respectively the two different Φs listed as Φ_4 and Φ_5 respectively.

Π_5: 1 8 12 13, 2 7 11 14 give differences 7′ 11 12* 4* 5′ 1, 5′ 9 12* 4* 7′ 3. The necessary differences 12*, 4* yield Θ_3, and the *two* sets of fortuitous matching differences given by the dashed 7s and 5s yield the two different Φs listed as Φ_6 and Φ_7 respectively.

Π_6: 1 4 14 15, 5 8 10 11 give differences 3*′ 13 14 10 11 1*, 3* 5 6 2 3′ 1. The necessary differences 3*, 1* yield the same Θ_3 as above (as they must) and the fortuitous matching differences given by the dashed 3s yield Φ_3 as listed.

(*d*) *Summary of enumerations*

This completes the process. It confirms the existence of the three (only) Frénicle quadsets of genus Θ, and establishes that there are seven Frénicle quadsets of genus Φ, namely those listed: none from Π_1, one from each of Π_2, Π_3, Π_6, and two from each of Π_4, Π_5.

This completes the enumeration and identification of all Frénicle quadsets. The complete list gives Π: 6; Θ: 3; Φ: 7; X: 14; Λ: 0; Ω: 4, making 34 in all. This total of 34, which is also the sum of the numbers in all rows, columns, principal diagonals and Frénicle quadsets of solutions, appears to be coincidence, that is, among the many curious facts associated with these magic squares.

The total number of *regular* solutions is thus $48\times6+80\times3+16\times7+16\times14 = 864$, which together with $4\times4 = 16$ *irregular* solutions of genus Ω give the total 880 for all essentially different solutions.

The proof of Frénicle's result by the part-sum method is now complete. It has required no consideration of actual solutions, but only their structure, and is therefore strictly an analytical proof. The main interest in these solutions undoubtedly lies, not so much in the truly magical properties of the 48 pandiagonal squares, for one would expect symmetries, but in the extraoridinary fact of there being any irregular solution, and then merely 16 such irregulars. The method of proof given here reveals why this is so. Had there been no irregular solutions, with a total of (regular) solutions then $864 = 2^3\times3^3$ it might not have been a matter of much surprise. But that, from all the $16!/32 = (6.538\ldots)^{11}$ possible permutations of the first sixteen numbers to form essentially different squares of which $880\times8 = 7040$ are magic, exactly $16\times8 = 128$ *and no more* are 'irregular' is certainly surprising. One could reasonably conjecture that there could be huge numbers of irregular solutions (the number of 5×5 magic squares is enormous and has only recently been computed), and indeed, had solutions with Frénicle quadsets of genus Λ and/or of genus Ω with no unrepeated part sums been possible, the total number of solutions would have been as much as doubled. It appears to be mere chance, arising from the intriguing and often perverse behaviour of the positive integers, which gives such a happy and manageable number of solutions. This should not lessen our admiration for Bernard de Frénicle's achievement in taking this chance and successfully working out 'by hand' the correct answer so long ago.

5. The alternative 'matrix method'

Consider now the magic squares constructed from the numbers 0–15 and expressed in scale 4, so that each position in the squares contains two digits from among the sixteen different ordered pairings

00	01	02	03
10	11	12	13
20	21	22	23
30	31	32	33

of the 'normal array'. As has been said earlier, a first condition for a solution is that each of these pairings occurs once and only once, with each digit 0 1 2 3 occurring four times in the first or 'radix' position and four times in the second or 'unit' position. The square can thus be thought of as the combination (or superposition) of two 4×4 single-digit matrices each of which is made up of four each of the digits 0 1 2 3 in some arrangement. Call these respectively the radix matrix and the unit matrix.

In the 0–15 magic square, by definition, the numbers in the rows, in the columns and in the two principal diagonals must add to 30, as also, by Frénicle's rules, must the other 'defined quads', namely the short broken diagonals in semi-pandiagonal and pandiagonal squares and the long broken diagonals in pandiagonal squares. We have shown that all these 'defined quads' must be from the first or second groups of the quad list, and we notice that, in the quad list written in the 00–33 notation,

all quads in the first and second groups have digits in both the radix and the unit positions which are either 0 1 2 3 or 0 0 3 3 or 1 1 2 2 in some arrangement, a property which does not hold for any quad of the third group.

The only combinations of these three sets of four digits which can give 'digit quadsets' with each digit 0, 1, 2, 3, occurring exactly four times are

(i)	0 1 2 3	0 1 2 3	0 1 2 3	0 1 2 3
(ii)	0 1 2 3	0 1 2 3	0 0 3 3	1 1 2 2
(iii)	0 0 3 3	1 1 2 2	0 0 3 3	1 1 2 2

The principles on which the proof by the matrix method is based are as follows. To obtain the sum 30 for the magic square in its rows, columns, and principal diagonals, the sum of the digits of the radix matrix multiplied by 4 and added to the sum of the digits of the unit matrix must yield 30. The only possible combinations are:

$$30 = 7 \times 4 + 2 = 6 \times 4 + 6 = 5 \times 4 + 10.$$

It follows that the radix matrix of a successful combination must be such that the digits in any 'defined quad' of a digit quadset add to 7, 6 or 5; and that these defined quads must then be 'matched' in the unit matrix by quads the digits of which add respectively to 2, 6 or 10. A matrix therefore cannot be 'mixed', that is, it cannot contain a quad with digits adding to 7 or 5 *and* a quad with digits adding to 2 or 10. Moreover if a digit quadset contains a quad with digits adding to 7 then it must contain a pairing quad with digits adding to 5 and *vice-versa*; and if a digit quadset contains a quad with digits adding to 2 it must contain a pairing quad with digits adding to 10 and *vice-versa*. We call a unit (or radix) matrix *viable* if it is such that a 'matching' radix (or unit) matrix can be found with which it can potentially combine successfully in some orientation to form a solution, that is with which it can combine without causing duplication of numbers in the resulting square. If it can be shown that no such matching matrix can be found without necessarily causing duplications of numbers when superposed, it is *unviable* and can be ruled out from further consideration. The first stage of proof by the matrix method thus lies in establishing that all rows, all columns and all Frénicle quads in *both* the radix *and* the digit matrices (where Frénicle quads in these matrices are defined as in solutions) of potentially viable matrices must have digits 0 1 2 3, 0 0 3 3, or 1 1 2 2, a characteristic which we have now proved to be true by the part-sum method since it is proved in §2(*g*) that Frénicle quads must come from Groups 1 or 2 of the quad list. There are 35 different ways of choosing four digits (unordered) from four digits 0 1 2 3 using each not more than four times. Among these 35 are just 17 which have digits which add to 2, 5, 6, 7 or 10. They are shown in the table below.

Totals of four digits

2	5	6	7	10
0 0 1 1	0 0 2 3	0 0 3 3	3 3 1 0	3 3 2 2
0 0 0 2	0 1 1 3	0 1 2 3	3 2 2 0	3 3 3 1
	0 1 2 2	0 1 2 2	3 2 1 1	
	1 1 1 2	1 1 1 3	2 2 2 1	
		0 2 2 2		

Here the sets of four digits 0 1 2 3, 0 0 3 3, 1 1 2 2 used in various ways form the radix and the unit digit respectively of all quads of the first and second groups in the 00–33 quad list (see Appendix I), whereas all sets of four digits shown above other than 0 0 3 3, 1 1 2 2 are required to describe completely the radix and unit digits of the quads in the third group of the quad list.

There are eleven different ways of combining the digit quads with sums 2, 6 or 10 into quadsets which contain each digit four times and which could thus form quadsets in a unit matrix, three of these being the quadsets (i), (ii), (iii) already mentioned. Six of the remaining eight can be readily dismissed as leading to unviable unit matrices which could not be 'matched' by any radix matrix without causing duplications. The principal difficulty lies in eliminating the possibility of the quadsets

	0 2 2 2	1 1 1 3	0 0 0 2	1 3 3 3,
and	0 0 1 1	2 2 3 3	0 0 1 1	2 2 3 3,

which, when we call the digits 0, 3 and 1, 2 complementary, are self-complementary digit quadsets.

With this achieved (and now proved by the part-sum method), it becomes a simple matter to write down all possible potentially viable matrices and thus, on following certain guidelines, to construct all possible magic squares. The matrices are listed in the appendices, ordered and labelled in the manner in which they were arrived at originally by this method and from which the symmetrical solution list was constructed *before any theoretical work on, or thought of, providing proof*. That the matrices should obey the rules described above was merely an obvious guess and a way of constructing them; that they combined successfully to form a list of exactly 880 solutions was sufficient reassurance that the matrix list was surely correct and exhaustive, and gave rise to an irresistible encouragement to attempt rigorous proof.

In order to keep to a standard orientation of matrices we write them so that when the digits 0 0 3 3, 1 1 2 2 may occur in *either* rows *or* columns, then they are arranged to do so in the rows. The permissible digit quadsets (i), (ii), (iii) make plain that the columns then have to be of the form (i), for no column composed of the digits 0 0 3 3 and 1 1 2 2 can intersect *two* rows which have the digits 0 0 3 3 and 1 1 2 2 respectively. In the standard form we can then state that all columns in any viable matrix are composed of the digits 0 1 2 3.

With the matrices written in the standard form we can attach to them the appropriate pattern labels (1) to (12), analogous to the patterns for solutions whenever they occur, formed by linking the complementary digits 0, 3 and 1, 2, the patterns here being read either 'vertically' or 'horizontally'. Patterns for matrices will not necessarily be unique in the way that they must be unique for solutions. However, it is plain that a solution with a particular pattern, when broken down to reveal its two contributing matrices, will have imparted its patterns to the matrices in some orientation. In other words, a necessary (but not sufficient) condition for two matrices to be able to combine successfully to form a solution is that in some orientation they have *one and at most one* pattern in common, since, if more than one pattern were to coincide between the unit and radix matrices, there would necessarily be duplication of numbers and so no solution.

It is plain that viable matrices must remain viable (in some orientation) under rotation, reflexion, and the transforms U, S. Viable matrices will always be Latin squares because the conditions imposed, as shown, on their rows and columns ensure that the sums of their components are always equal (to 6) and this is sufficient condition for a *Latin* square. The digits in their principal diagonals (which also form two Frénicle digit quads and must therefore together add to 12) may however be the two sets 0 2 2 2, 3 1 1 1; or they may add respectively to 7 and 5, or to 2 and 10. Whenever a solution is formed by the combination of two matrices which have diagonals with digits which add to six (including here therefore those with digits 0 2 2 2, 3 1 1 1), since the digits in the rows and columns and Frénicle quads of both matrices also always add to six, then reversing the digits in the radix and unit positions of each number in a solution will also give a solution, although not necessarily a solution not already looked after by the transformations U, S, US. This condition thus defines solutions which are 'reversible'. When the digits in the radix and unit poisitions in the principal diagonals add to 7 and 5 or to 2 and 10 respectively thus giving (when correctly positioned) correct diagonals in the solution, the reversal still gives a Latin square, but not a magic square, for its diagonals will now have the sums $10 \times 4 + 5 = 45$ and $2 \times 4 + 7 = 15$ respectively instead of both summing to 30.

There are 114 essentially different viable matrices in all and they have been grouped into three categories defined by the nature of their diagonals. Category One contains all matrices with diagonals formed by sets of digits 0 1 2 3, 0 0 3 3, or 1 1 2 2; Category Two has matrices with diagonals 0 2 2 2, 3 1 1 1; Category Three has matrices whose diagonals have digits which add to 7 and 5, or to 2 and 10 as explained. The '7, 5' matrices can combine only with the '2, 10' matrices and then only in positions which give 'matching' diagonals and with the '7, 5' matrix in the radix or left-hand position and the '2, 10' matrix in the unit or right-hand position. The matrices with diagonals 0 2 2 2, 3 1 1 1 cannot combine with themselves since this would lead to duplication of numbers in the diagonals of the resulting squares. They can thus combine only with matrices of the first category which have diagonals composed of the digits 0 1 2 3.

The matrices are given in full in the List of Matrices (Appendix VI) where they are also given their appropriate pattern numbers. The way they are grouped (and *named*, with roman capitals) is a result of the logical manner in which they were written in the first instance. The matrices A, B, C (and their transformations A′, B′, C′ by U) are the only matrices for which 'like digits' occur once and only once in each row, column and principal diagonal. Their 144 combinations which give essentially different successful magic squares (the other 144 combinations each leading to eight repeated numbers) necessarily lead to the only solutions to the magic card problem as will be explained in §6. Not only do 'like digits' form the pattern shown by the symbols ● in some rotation of reflection

●	•	•	•
•	•	•	●
•	●	•	•
•	•	●	•

when mapped on the normal array now, in digit form

$\underline{0}$	1	2	3
0	1	2	$\underline{3}$
0	$\underline{1}$	2	3
0	1	$\underline{2}$	3

or its rotations or transformations, but so also do sets of digits 0 1 2 3 within each matrix, as indeed they must for a successful combination.

The grouping (and naming) of other matrices is such that if any two 'lead' matrices combine successfully, then so also must all other pairs of matrices in 'matching' positions and orientations within the group. It is this property of the arrangement of the matrices which leads directly to the symmetries and order of the solution list.

Other guides to the successful combination of matrices are: within Category One, the matrices P, P′, Q, Q′, R, R′, T, T′ (which lead to the solutions with pattern (3) whose Frénicle quadsets are of genus Θ but are yet not semi-pandiagonal) can combine only between themselves and in restricted ways, this being caused by the positions of like digits within them and the positions of 'matching' sets of digits 0 1 2 3; the matrices J, M, M′, K, N, N′ cannot combine between themselves because of their high degree of symmetry and the nature of diagonals which are composed of the digits 0 0 3 3, 1 1 2 2 respectively. Matrices J and K can however combine successfully in several orientations with E, E′; and J, M, M′, K, N, N′ combine successfully in various ways with the group of matrices H, H′, I, I′.

The most powerful guide to both elimination of potentially successful combinations of matrices and to their likely success is the pattern numberings. Once the obvious exclusions mentioned above have been made we can look for like pattern numbers. If two matrices are superposed in any orientations which result in there being *two* common patterns read either horizontally or vertically, then as has been explained they cannot combine without causing duplications, for a square adopts common patterns of contributing matrices and a solution must always have a *unique* pattern. In contrast, whenever two matrices whose successful combination has not otherwise been excluded can be brought together so that they have one and only one common pattern then the result will almost always be a solution – and then a solution with this as its pattern. By following these and certain other simple rules, all possibilities for successful combinations of matrices can readily be established. However, the *theoretical* enumeration by this method of how many successful combinations there are is tedious and, as the Frénicle quad and part sum method has proved so powerful and effective in this respect, this has not here been further pursued.

The matrix method does however provide an extremely practical and speedy way of producing the list of solutions, having the added advantage of giving a highly symmetrical list of solutions which proved to be of great use in all the theoretical work which followed.

6. The magic card and the boss problems

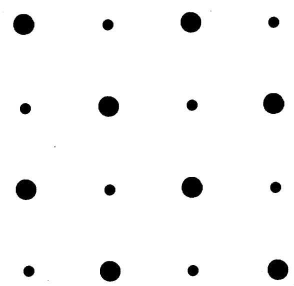
The magic card problem

That the 16 court cards of a pack can be placed in a 4×4 array so that no row, no column and neither of the two principal diagonals contains more than one card of each suit nor more than one of each rank in 144 essentially different ways can perhaps most easily now be shown by considering matrices. Designate cards of rank Ace, King, Queen, Jack by $r(u)$, $u = 0, 1, 2, 3$ respectively, and cards belonging to suits Spades, Hearts, Diamonds, Clubs by $s(v)$, $v = 0, 1, 2, 3$ respectively, so that each of the 16 court cards is uniquely defined by the different combinations of $r(u)+s(v)$. Then solutions to the magic card problem must be superpositions of two 4×4 matrices, both being such that one and only one digit from each of the four digits 0, 1, 2, 3 appears in each row, each column and in each of the two principal diagonals, the digits in each of these lines thus adding to six. If the matrix representing the ranks is thought of as being in the first or radix position and that representing the suits as being in the second or unit position (or vice versa) the sums of the numbers in each line in scale four 00–33 notation will thus be $6\times 4+6 = 30$, and the solution must form a magic square. The only matrices satisfying these conditions are those labelled A, A′, B, B′, C, C′, in the matrix list. There are 144 essentially different superpositions of these six matrices in orientations which give all 16 numbers with therefore no duplications. They are to be found in the list of magic square solutions numbered as shown in the table below:

pattern	solution numbers				
(4) (5)	1–8	49–56	97–104	145–152	32
(1) (2)	17–24	65–72	113–120	161–168	32
(1) (2)	33–40	81–88	129–136	176–184	32
(6)	289–292	329–332	309–372	409–412	16
(3)	297–300	337–340	377–388	417–420	16
(3)	313–316	353–356	393–396	433–436	16
					total 144

The boss problem

This problem is to determine how many 4×4 squares can be successfully arrived at with the normal boss described in §1, the space being taken as the missing number 16. The nature of the quads in the first and second groups of the quad list shows that every solution of the set of 880 can be obtained by an even number of interchanges from numbers arranged in the normal boss array – a simple check being that all four members of any of these quads can be brought into any particular row by an even number of orthogonal moves. Ball demonstrates that turning a square through 90° involves an odd number of interchanges, and so an arrangement which cannot be achieved in its initial orientation can be achieved when rotated through a right angle and vice versa. It follows that each essentially different 4×4 magic square gives rise, by rotation and reflection, to four which can be achieved and four which cannot be achieved with a boss of either the normal or the alternative form which is its 'reversal'. But, as is clear from what has been said above, all 880 solutions *written with TOP as in the pattern diagrams*, that is, in our defined 'standard form', can be obtained from the normal boss array by an even number of interchanges. We need, therefore, only to count in how many solutions the number 16 (33) occurs an even number of orthogonal moves from the bottom right-hand corner (the space in the normal boss); that is, in how many of the 880 solutions, as written in the 00–33 solution list, the number 33 lies in one or other of the positions shown as ● in the diagram below:



Actual counting is not necessary. If in a 'lead' solution, 33 lies on a principal diagonal, then the transforms U, S leave its parity unchanged. If it does not lie on a principal diagonal but occupies a 'side' position, whereas the transform S still leaves its parity unchanged, the U transform changes the parity. We have therefore only to take note of the parities of the number 33 in each lead solution and multiply by suitable factors according to whether it lies in a principal diagonal or occupies a 'side' position within the square.

We can quickly check that there are 104 lead solutions with 33 on a principal diagonal, 60 of which are in the top-left to bottom-right diagonal and 44 in the top-right to bottom-left diagonal. The other 116 lead solutions have the number 33 in side positions. The total number of solutions 'in standard orientation' for which the boss problem can be solved is thus $\{60+\frac{1}{2}(116)\}\times 4 = 472$, the number for which it fails being $\{44+\frac{1}{2}(116)\}\times 4 = 408$. The difference is accounted for by the way in which the solutions with pattern (3) behave under the transformations U, S.

7. Summary of significant or curious facts (not all of which have occurred in the earlier parts of the paper)

(1) There are 880 essentially different solutions (Frénicle, pre-1675) which can be directly derived from 220 essentially different solutions by transformations U, S (Lehmer 1933).

(2) In every solution the rows, the columns and the Frénicle quads (§3(*d*)) contain respectively two elements which have the same sum (a 'common part sum'); in some solutions these quads may have two (overlapping) pairs of two numbers with the same sum.

(3) All rows, columns and Frénicle quads (which must have elements adding to 34) contain two odd and two even numbers. The two principal diagonals (which also must have elements adding to 34 by definition) can, in contrast, contain all-odd and all-even numbers respectively.

(4) The links between complementary numbers within any solution form one of twelve patterns (Andrews 1908, Dudeney 1910, both by observation only). Ten of these patterns are symmetrical about both the vertical and horizontal axes of the square, giving 864 *regular* solutions; the other two patterns are symmetrical about one axis only and giving the 16 *irregular* solutions.

(5) There are 86 sets of four numbers which can be chosen from the numbers 1 to 16 which add to 34 and are defined here as quads. Of these, 28 are self complementary and can be combined in 105 essentially different ways to give four compatible quads (quadsets) which together contain each number 1–16 once and only once. Another 24 quads form 12 mutually-complementary pairs which can be combined to form *six* quadsets (only), these being the combination of two mutually-complementary pairs of quads and such that each quad appears once and only once in the six quadsets taken together. The remaining 34 quads form a further group of 17 mutually-complementary pairs, *no two pairs forming a compatible quadset with any other mutually-complementary pair.*

(6) The self-complementary quads necessarily contain two odd and two even numbers. The second group of 24 quads also are such that each contains two odd and two even numbers. If the numbers 1–16 are 'converted' into the numbers 0–15 by subtracting 1 and then expressing them in the 00–33 scale 4 notation, these 52 quads all have the characteristic that the digits in the radix positions *and* the digits in the unit positions respectively of the four numbers in each quad are either 0 1 2 3, 0 0 3 3 or 1 1 2 2 in some arrangement. This property does *not* hold for any of the 34 quads in the third group.

(7) The 28 self-complementary quads are all 'used' either as rows (or columns) or as Frénicle quads. Of the 105 quadsets into which they can be combined, *three* are used as either rows (or columns) or as Frénicle quadsets (necessarily in different solutions), *seven* are used as Frénicle quadsets but never as rows (or columns) and 24 are used as rows (or columns) but never as Frénicle quadsets. Thus of the 105 'all self-complementary quadsets' 34 in all are 'used'.

(8) The six quadsets formed by the 12 mutually-complementary pairs of quads are all used as rows, as columns and/or as Frénicle quadsets. In 96 solutions (48 defined as *pandiagonal* and 48 defined as 'diagonal') they are used in combination simultaneously as either rows, and/or columns and/or Frénicle quadsets. In pandiagonal and semi-pandiagonal solutions the rows, the columns, the Frénicle quadsets and the quadsets formed by principal and short-broken diagonals are all from among the nine quadsets Π_{1-6}, Θ_{1-3} in some combination.

(9) There are 34 essentially different Frénicle quadsets, two pairs (only) of which are complements. This frequent recurrence of the number 34 appears to be mere coincidence.

(10) The common part sum or sums of any set of four quads forming a quadset must be 9, 13, 15, 16 or their complements, or 17. Those pairs of numbers adding to 9, 13, 15, 16 form a common part sum must be specific *constructive part sums* (§2(*g*)) namely 1+8, 2+7, 3+6, 4+5; 1+12, 2+11, 3+10, 4+9; 1+14, 2+13, 3+12, 4+11; 1+15, 2+14, 3+13, 4+12. Other pairs of numbers adding respectively to 13, 15, 16, although of significance in determining possible solutions, are *non-constructive*. There are no non-constructive pairs of numbers adding to 9.

(11) Since all rows, columns and Frénicle quads have two even and two odd numbers, the three part sums of any such 'defined quads' must be two odd and one even (§2(*f*)).

(12) In no solution can the rows, the columns or the Frénicle quadsets be composed of one self-complementary quad and three quads with no complementarity. Other classifications of

quadsets (§2(*d*)) all occur as rows, columns, and/or Frénicle quadsets somewhere among the 880 solutions.

(13) The '*triad*' (α, β, γ) of part sums results from the quad with elements $\frac{1}{2}(\alpha+\beta-\gamma)$, $\frac{1}{2}(\alpha-\beta+\gamma)$, $\frac{1}{2}(-\alpha+\beta+\gamma)$, $34-\frac{1}{2}(\alpha+\beta+\gamma)$ or from its complement. The element 16 (and its complement 1) can only result from the last expression and then only if $\alpha+\beta+\gamma = 36$. A set of four triads is only admissible if two triads add to 36 (the corresponding quads containing either 1 *or* 16) *and* if one of the following alternatives apply (not proved here): (i) if neither of the 'sum-36' triads contains the part sum 16, then each of the other two triads' sums must be 38; (ii) if just one of the 'sum-36' triads contains the part sum 16, one of the triads with sum greater than 36 must have the sum 38 and the other a sum not less than 40; (iii) if both the 'sum-36' triads contain the part sum 16, then each of the other two must have a sum not less than 40, this arising from consideration of elements 2 and 15.

(14) All magic squares can be obtained from one another by an even number of interchanges (§6). There are 144 essentially different solutions to the magic card problem. In the 'normal' boss game 472 magic squares written in the 'standard form' can be achieved, the other 408 magic squares being unachievable when written in the standard form.

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References

Andrews, W. S. 1908, 1917 *Magic squares and cubes*. Chicago: Open court Pub. Co., pp. 119.
Ball, W. W. R. (revised Coxeter, H. S. M.) 1944 *Mathematical recreations and essays*, pp. 193–221. London: Macmillan.
Benson, W. H. & Jacoby, O. 1976 *New recreations with magic squares* (198 pages). New York: Dover Publications.
Bergolt, E. 1910 *Nature, Lond.* **83**, 368–369.
Boyer, C. B. 1968 *A history of mathematics*, p. 219. London: John Wiley & Sons.
Dudeney, H. E. Jan. 15, 1910 The magic square of sixteen. *The Queen, The Lady's Newspaper, London*, pp. 125–126.
Dudeney, H. E. 1917 *Amusements in mathematics*, pp. 119–126. London: T. Nelson & Sons.
Frénicle, B. de Bessey 1693, 1731 (both posthumous) Des Quarrez ou Tables Magique: Les Memoires de L'Academie des Sciences (The Hague).
Gardner, M. 1976 *Sci. Am.* **234**, 118–122.
Kraitnik, M. 1938 *Mathematical recreations*, pp. 182–192. London: George Allen and Unwin.
Lehmer, D. N. 1933 A Census of Squares of Order Four. *Bull. Am. Math. Soc.* **39**, p. 981.
Rosser, B. & Walker, R. J. 1938 *Bull. Am. Math. Soc.* **44**, 416–420.
Schaaf, W. L. 1970, 1973, 1978 *A bibliography of recreational mathematics*, vols 1, 2, 3, 4. National Council of Teachers of Mathematics 1201 Sixteenth Street, Northwest, Washington, D.C. 20036.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
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APPENDICES

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Appendix I

(i) List of all quads in 00–33 scale 4 notation

Group 1. Self-complementary quads

00	01	32	33	01	02	31	32	02	03	30	31	03	10	23	30
00	02	31	33	01	03	30	32	02	10	23	31	03	11	22	30
00	03	30	33	01	10	23	32	02	11	22	31	03	12	21	30
00	10	23	33	01	11	22	32	02	12	21	31	03	13	20	30
00	11	22	33	01	12	21	32	02	13	20	31	10	13	20	23
00	12	21	33	01	13	20	32	11	13	20	22	10	12	21	23
00	13	20	33	12	13	20	21	11	12	21	22	10	11	22	23

Group 2. Mutually-complementary pairs which combine to form quadsets

00	03	31	32	01	02	30	33	10	13	21	22	11	12	20	23
00	11	23	32	01	10	22	33	02	13	21	30	03	12	20	31
00	13	21	32	01	12	20	33	02	11	23	30	03	10	22	31
00	12	23	31	02	10	21	33	01	13	22	30	03	11	20	32
00	13	22	31	02	11	20	33	01	12	13	30	03	10	21	32
00	13	23	30	03	10	20	33	01	12	22	31	02	11	21	32

Group 3. Mutually-complementary pairs which do not combine

00	10	30	32	01	03	23	33
00	12	22	32	01	11	21	33
00	11	30	31	02	03	22	33
00	20	21	31	02	12	13	33
00	20	22	30	03	11	13	33
00	21	22	23	10	11	12	33
01	10	30	31	02	03	23	32
01	11	23	31	02	10	22	32
01	13	21	31	02	12	20	32
01	20	21	30	03	12	13	32
01	20	22	23	10	11	13	32
02	12	22	30	03	11	21	31
02	13	22	23	10	11	20	31
02	20	21	23	10	12	13	31
03	12	22	23	10	11	21	30
03	13	21	23	10	12	20	30
03	20	21	22	11	12	13	30

(ii) List of all quads in 1–16 notation

Group 1. Self-complementary quads

1	2	15	16	2	3	14	15	3	4	13	14	4	5	12	13
1	3	14	16	2	4	13	15	3	5	12	14	4	6	11	13
1	4	13	16	2	5	12	15	3	6	11	14	4	7	10	13
1	5	12	16	2	6	11	15	3	7	10	14	4	8	9	13
1	6	11	16	2	7	10	15	3	8	9	14	5	8	9	12
1	7	10	16	2	8	9	15	6	8	9	11	5	7	10	12
1	8	9	16	7	8	9	10	6	7	10	11	5	6	11	12

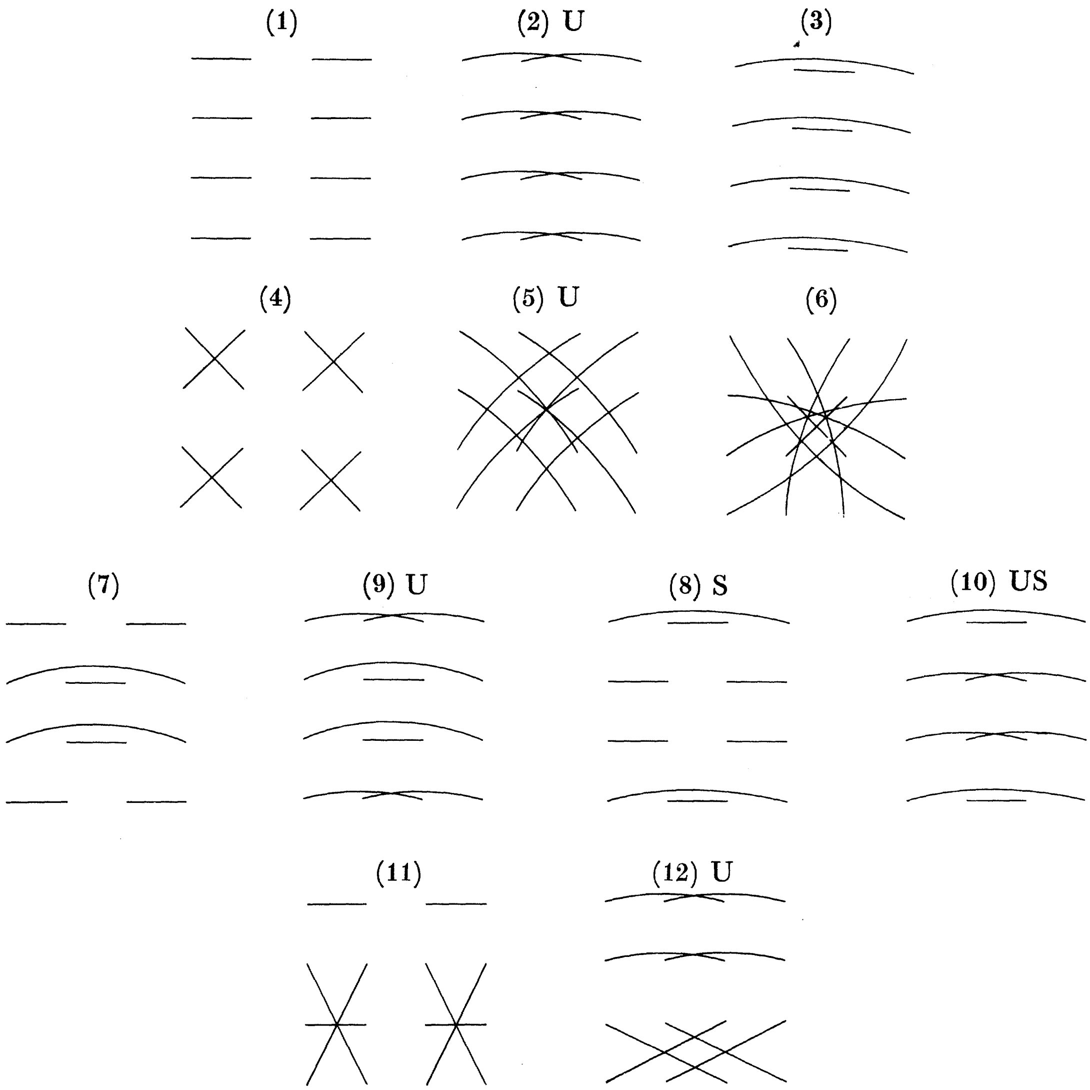
Group 2. Mutually-complementary pairs which combine to form quadsets

1	4	14	15	2	3	13	16	5	8	10	11	6	7	9	12
1	6	12	15	2	5	11	16	3	8	10	13	4	7	9	14
1	8	10	15	2	7	9	16	3	6	12	13	4	5	11	14
1	7	12	14	3	5	10	16	2	8	11	13	4	6	9	15
1	8	11	14	3	6	9	16	2	7	12	13	4	5	9	15
1	8	12	13	4	5	9	16	2	7	11	14	3	6	10	15

Group 3. Mutually-complementary pairs which do not combine

1	5	13	15	2	4	12	16
1	7	11	15	2	6	10	16
1	6	13	14	3	4	11	16
1	9	10	14	3	7	8	16
1	9	11	13	4	6	8	16
1	10	11	12	5	6	7	16
2	5	13	14	3	4	12	15
2	6	12	14	3	5	11	15
2	8	10	14	3	7	9	15
2	9	10	13	4	7	8	15
2	9	11	12	5	6	8	15
3	7	11	13	4	6	10	14
3	8	11	12	5	6	9	14
3	9	10	12	5	7	8	14
4	7	11	12	5	6	10	13
4	8	10	12	5	7	9	13
4	9	10	11	6	7	8	13

Appendix II. The twelve solution patterns



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Appendix III. The quad arrays

Common part sum 9

1 8 9 16 (*9* 10 17)	1 8 10 15 (*9* 11 *16*)	1 8 11 14 (*9* 12 *15*)	1 8 12 13 (*9* 13 *14*) ;
2 7 9 16 (*9* 11 *16*)	2 7 10 15 (*9* 12 17)	2 7 11 14 (*9* *13* 16)	2 7 12 13 (*9* 14 *15*) ;
3 6 9 16 (*9* 12 *15*)	3 6 10 15 (*9* *13* 16)	3 6 11 14 (*9* 14 17)	3 6 12 13 (*9* 15 *16*) ;
4 5 9 16 (*9* *13* 14)	4 5 10 15 (*9* 14 *15*)	4 5 11 14 (*9* 15 *16*)	4 5 12 13 (*9* 16 17) .

Common part sum 13

1 5 12 16 (6 *13* 17)	1 6 12 15 (7 *13* *16*)	1 7 12 14 (8 *13* *15*)	1 8 12 13 (*9* *13* 14) ;
2 5 11 16 (7 *13* *16*)	2 6 11 15 (8 *13* 17)	2 7 11 14 (*9* *13* 16)	2 8 11 13 (10 *13* *15*) ;
3 5 10 16 (8 *13* *15*)	3 6 10 15 (*9* *13* 16)	3 7 10 14 (10 *13* *17*)	3 8 10 13 (11 *13* *16*) ;
4 5 9 16 (*9* *13* 14)	4 6 9 15 (10 *13* *15*)	4 7 9 14 (11 *13* *16*)	4 8 9 13 (12 *13* 17) .

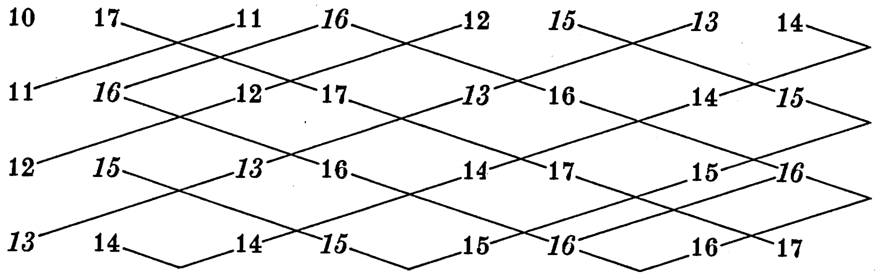
Common part sum 15

1 3 14 16 (4 *15* 17)	1 7 12 14 (8 *13* *15*)	1 8 11 14 (*9* 12 *15*)	1 4 14 15 (5 *15* *16*) ;
3 5 10 16 (8 *13* *15*)	5 7 10 12 (12 *15* 17)	5 8 10 11 (13 *15* *16*)	4 5 10 15 (*9* 14 *15*) ;
3 6 9 16 (*9* 12 *15*)	6 7 9 12 (13 *15* *16*)	6 8 9 11 (14 *15* 17)	4 6 9 15 (10 *13* *15*) ;
2 3 13 16 (5 *15* *16*)	2 7 12 13 (*9* 14 *15*)	2 8 11 13 (10 *13* *15*)	2 4 13 15 (*6* *15* 17) .

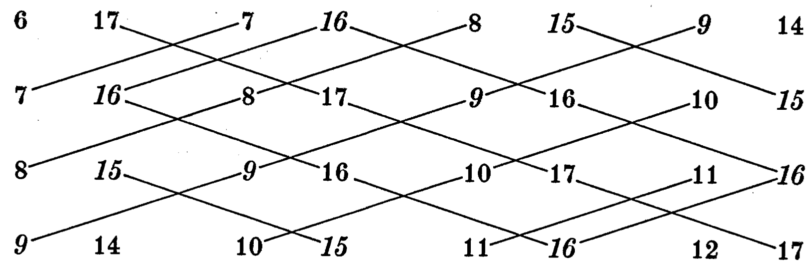
Common part sum 16

1 2 15 16 (3 *16* 17)	1 6 12 15 (7 *13* *16*)	1 8 10 15 (*9* 11 *16*)	1 4 14 15 (5 *15* *16*) ;
2 5 11 16 (7 *13* *16*)	5 6 11 12 (11 *16* 17)	5 8 10 11 (13 *15* *16*)	4 5 11 14 (*9* 15 *16*) ;
2 7 9 16 (*9* 11 *16*)	6 7 9 12 (13 *15* *16*)	7 8 9 10 (15 *16* 17)	4 7 9 14 (11 *13* *16*) ;
2 3 13 16 (5 *15* *16*)	3 6 12 13 (*9* 15 *16*)	3 8 10 13 (11 *13* *16*)	3 4 13 14 (7 *16* 17) .

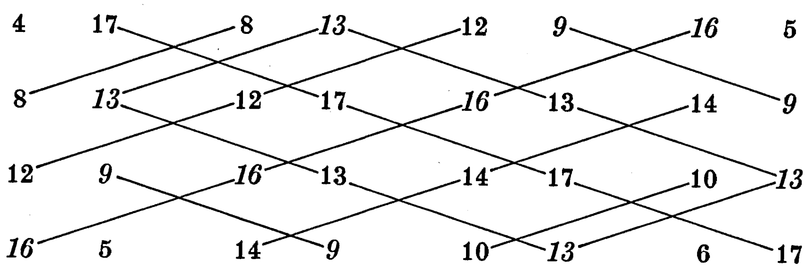
Common part sum 9



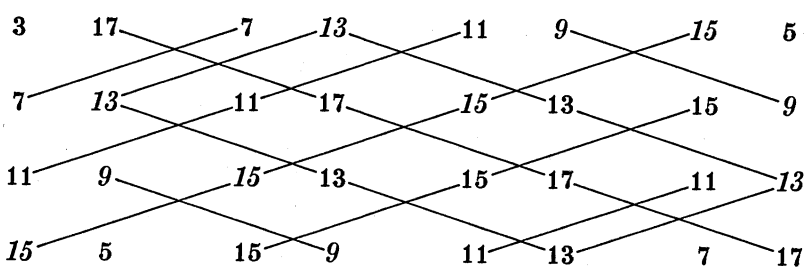
Common part sum 13



Common part sum 15



Common part sum 16



The part sums which are in italic type are 'constructive part sums'. There are part sums 13, 15, 16 which are not constructive. All part sums 9 are necessarily constructive.

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Appendix IV. Frénicle quadsets (with part sums)

genus Π				
1.	1 8 11 14	2 7 12 13	3 6 9 16	4 5 10 15
	(*9* 12 *15*)	(*9* 14 *15*)	(*9* 12 *15*)	(*9* 14 *15*);
2.	1 6 12 15	3 8 10 13	2 5 11 16	4 7 9 14
	(7 *13* *16*)	(11 *13* *16*)	(7 *13* *16*)	(11 *13* *16*);
3.	1 8 10 15	3 6 12 13	2 7 9 16	4 5 11 14
	(*9* 11 *16*)	(*9* 15 *16*)	(*9* 11 *16*)	(*9* 15 *16*);
4.	1 7 12 14	2 8 11 13	3 5 10 16	4 6 9 15
	(8 *13* *15*)	(10 *13* *15*)	(8 *13* *15*)	(10 *13* *15*);
5.	1 8 12 13	2 7 11 14	4 5 9 16	3 6 10 15
	(*9* *13* 14)	(*9* *13* 16)	(*9* *13* 14)	(*9* *13* 16);
6.	1 4 14 15	5 8 10 11	2 3 13 16	6 7 9 12
	(5 *15* *16*)	(13 *15* *16*)	(5 *15* *16*)	(13 *15* *16*).
genus Θ				
1.	1 6 11 16	2 5 12 15	3 8 9 14	4 7 10 13
	(7 12 17)	(7 14 17)	(11 12 17)	(11 14 17);
2.	1 7 10 16	2 8 9 15	3 5 12 14	4 6 11 13
	(8 11 17)	(10 11 17)	(8 15 17)	(10 15 17);
3.	1 4 13 16	2 3 14 15	5 8 9 12	6 7 10 11
	(5 14 17)	(5 16 17)	(13 14 17)	(13 16 17).
genus Φ				
1.	1 6 11 16	4 8 9 13	3 7 10 14	2 5 12 15
	(7 12 17)	(12 *13* 17)	(10 *13* 17)	(7 14 17);
2.	1 8 9 16	4 6 11 13	3 5 12 14	2 7 10 15
	(*9* 10 17)	(10 15 17)	(8 15 17)	(*9* 12 17);
3.	1 4 13 16	6 8 9 11	5 7 10 12	2 3 14 15
	(5 14 17)	(14 *15* 17)	(12 *15* 17)	(5 16 17);
4.	1 5 12 16	2 4 13 15	6 8 9 11	3 7 10 14
	(6 *13* 17)	(6 *15* 17)	(14 *15* 17)	(10 *13* 17);
5.	5 7 10 12	4 8 9 13	2 6 11 15	1 3 14 16
	(12 *15* 17)	(12 *13* 17)	(8 *13* 17)	(4 *15* 17);
6.	1 8 9 16	3 7 10 14	2 6 11 15	4 5 12 13
	(*9* 10 17)	(10 *13* 17)	(8 *13* 17)	(*9* 16 17);
7.	2 7 10 15	4 8 9 13	1 5 12 16	3 6 11 14
	(*9* 12 17)	(12 *13* 17)	(6 *13* 17)	(*9* 14 17).

genus X

1.	1 8 10 15 (*9* 11 *16*)	5 6 11 12 (11 *16* 17)	2 7 9 16 (*9* 11 16)	3 4 13 14 (7 *16* 17);
2.	3 6 12 13 (*9* 15 *16*)	7 8 9 10 (15 *16* 17)	4 5 11 14 (*9* 15 *16*)	1 2 15 16 (3 *16* 17);
3.	1 7 12 14 (8 *13* *15*)	2 6 11 15 (8 *13* 17)	3 5 10 16 (8 *13* *15*)	4 8 9 13 (12 *13* 17);
4.	2 8 11 13 (10 *13* *15*)	3 7 10 14 (10 *13* 17)	4 6 9 15 (10 *13* *15*)	1 5 12 16 (6 *13* 17);
5.	1 8 11 14 (*9* 12 *15*)	5 7 10 12 (12 *15* 17)	3 6 9 16 (*9* 12 *15*)	2 4 13 15 (6 *15* 17);
6.	2 7 12 13 (*9* 14 *15*)	6 8 9 11 (14 *15* 17)	4 5 10 15 (*9* 14 *15*)	1 3 14 16 (4 *15* 17);
7.	1 8 11 14 (*9* 12 *15*)	2 7 10 15 (*9* 12 17)	3 6 9 16 (*9* 12 *15*)	4 5 12 13 (*9* 16 17);
8.	2 7 12 13 (*9* 14 *15*)	3 6 11 14 (*9* 14 17)	4 5 10 15 (*9* 14 *15*)	1 8 9 16 (*9* 10 17);
9.	1 6 12 15 (7 *13* *16*)	3 4 13 14 (7 *16* 17)	2 5 11 16 (7 *13* *16*)	7 8 9 10 (15 *16* 17);
10.	3 8 10 13 (11 *13* *16*)	5 6 11 12 (11 *16* 17)	4 7 9 14 (11 *13* *16*)	1 2 15 16 (3 *16* 17);
11.	1 8 12 13 (*9* *13* 14)	3 6 11 14 (*9* 14 17)	4 5 9 16 (*9* *13* 14)	2 7 10 15 (*9* 10 17);
12.	3 6 10 15 (*9* *13* 16)	4 5 12 13 (*9* 16 17)	2 7 11 14 (*9* *13* 16)	1 8 9 16 (*9* 10 17);
13.	1 8 10 15 (*9* 11 *16*)	4 5 12 13 (*9* 16 17)	2 7 9 16 (*9* 11 *16*)	3 6 11 14 (*9* 14 17);
14.	1 4 14 15 (5 *15* *16*)	7 8 9 10 (15 *16* 17)	2 3 13 16 (5 *15* *16*)	5 6 11 12 (11 *16* 17).

genus Ω

1.	1 8 12 13 (*9* *13* 14)	4 5 10 15 (*9* 14 *15*)	3 6 9 16 (*9* 12 *15*)	2 7 11 14 (*9* *13* 16);
2.	4 5 9 16 (*9* *13* 14)	2 7 12 13 (*9* 14 *15*)	1 8 11 14 (*9* 12 *15*)	3 6 10 15 (*9* *13* 16);
3.	1 8 10 15 (*9* 11 *16*)	4 7 9 14 (11 *13* *16*)	2 5 11 16 (7 *13* *16*)	3 6 12 13 (*9* 15 *16*);
4.	2 7 9 16 (*9* 11 *16*)	3 8 10 13 (11 *13* *16*)	1 6 12 15 (7 *13* *16*)	4 5 11 14 (*9* 15 *16*).

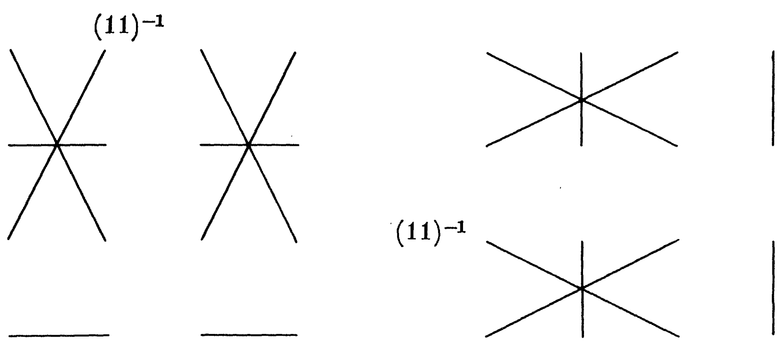
Appendix V. Notes on the presentation of the matrices and solutions

The matrices and the solutions are arranged in the same manner, this being governed in the first instance by the nature of their diagonals. The 30 matrices in Category One have diagonals with digits 0 1 2 3, 0 0 3 3, 1 1 2 2 respectively in some ordering and some combination. The eight matrices in Category Two have diagonals with digits 0 2 2 2, 1 1 1 3 respectively. The matrices of Category Three are of two kinds: 52 with diagonals having digits adding to 7, 5 respectively and 24 with diagonals having digits adding to 2, 10 respectively.

Category One matrices combine to give 528 essentially different Category One solutions with Frénicle quadsets of genus Π and genus Θ. The eight matrices of Category Two combine with eight 'matching' matrices of Category One to give 128 essentially different solutions of Category Two in four blocks of 32 solutions with Frénicle quadsets of genus X. The 52 Category Three (7, 5)–matrices combine with the 24 (2, 10)–matrices to give 224 Category Three solutions, 96 with Frénicle quadsets of genus X, 112 with Frénicle quadsets of genus Φ and 16 with Frénicle quadsets of genus Ω.

The matrices and the solutions are also labelled by their pattern designations. For *matrices*, the 'orthogonal patterns' (1), (2), (3), (7), (8), (9) (10) are always unique when a matrix is in the 'vertical orientation', that is, when it is viewed as though the left-hand column of digits as depicted were the top row of digits. This is because, as explained in §5, the matrices are oriented so that the columns have the digits 0 1 2 3 and the patterns for columns are thus uniquely determined by the positions of the 'complementary digits' 0 3, and 1 2. In the 'standard orientation' matrices can, and most do, have more than one orthogonal pattern. The symmetrical non-orthogonal patterns (4), (5), (6) must necessarily apply equally in both the horizontal and vertical orientations. These latter patterns occur only in Category One matrices (and in Category One solutions).

The patterns (11), (12), which are symmetrical in one median only, occur only in Category Three matrices. They are shown as $(11)^{-1}$, $(12)^{-1}$ when they occur 'upside down' or left-to-right as illustrated below for patterns (11)



instead of in the orientation shown in the pattern diagrams of the text and appendix. Pairs of 'knight's-move links' between complementary digits occur in some matrices, without necessarily being accompanied by the precise orthogonal links between the other complementary digits to give the full patterns (11), (12). These are labelled by these pattern numbers in square brackets i.e. as [11], [12], $[11]^{-1}$, $[12]^{-1}$ respectively as this then indicates the possible combinations to give patterns (11), (12) in solutions.

The pattern numbers in the solutions are necessarily always unique, as otherwise there would be duplicated numbers in the square.

In both the matrix and the solution lists the methods of obtaining matrices/solutions from one another are indicated in the same manner. The matrices/solutions in the second, third and fourth columns are respectively the results of operating the transformations U, S, US (as described) on matrices/solutions in the first column of the row. For solutions, these transformations always result in a row of four essentially different *solutions*. For matrices this does not necessarily hold and the positions are filled only when the resulting matrices are essentially different.

As explained, we call the top left-hand matrix/solution in any main block a 'lead' matrix/solution, where each block of solutions has the same Frénicle quadset and the same pattern or related patterns, i.e. (4), (5); (1), (2); (6); (3); (7), (9); (8) (10); (11), (12). Matrices/solutions in the principal left-hand columns within the blocks, other than the 'leads', are either marked by a small *f* (standing for the word *from*) followed by the index number of the matrix/solution from which they are derived by simple interchanges shown, or by a small *r* (standing for the word *reversal*), or by a small *c* (standing for the word *complement*). For some solutions either *c* or *r* could have been used. The letters *s-r* mean 'self reversing'.

The actual naming of the matrices by capital roman letters in Categories One and Two, and by arabic numerals and lower-case letters in the (7, 5) and (2, 10) groups of Category Three respectively, is largely the accident of how they were first written, *before* the full significance of transformations and groupings had been appreciated. For example, the first matrices to be written were the 'magic card' matrices labelled A, A′, B, B′, C, C′. The matrices P, P′, Q, Q′, R, R′, T, T′ (the letters S, U having been pre-empted by the two principal transformations) were 'slotted in' with the B, F and C, G blocks when the power of the block structures became apparent. The 'dash' indicates the transformation U. Thus U(A) = A′ and so on. This was a device to save letters in the labelling. Double dashes to indicate the transformation S would have been cumbersome. Thus new letters and numerals are used for transformations, e.g. $S(H) = I$, $S(1.) = 2.$, $S(a) = b$. Notice that the full stop is always used after the numerals labelling the (7, 5) matrices in order to avoid any possible confusion between these and the digits forming the matrices.

The *solutions* are numbered sequentially, but labelled also to indicate the two constituent matrices from which they are derived, the order of these being significant. The first matrix label indicates that this matrix is in the first or radix position; the second that the matrix is in the second or unit position. Solutions can have identical matrix labels and yet be essentially different when two matrices can be successfully combined in different orientations. For example, there are four solutions (namely, solutions 1, 3, 5, 7) labelled BB; and there are eight solutions, necessarily a maximum, labelled E′E, namely solutions 293, 295, 333, 335, 373, 375, 413, 415. This is because some matrices have special symmetries which can then be arranged in blocks in a variety of ways. The matrix B can be combined with itself successfully in four essentially different ways as can the matrix A, and the matrices E, E′ can be combined in sixteen essentially different ways to give solutions.

Because the transformation U changes an 'undashed' matrix label into a 'dashed' label and vice versa, in solutions which can be obtained from one another by the transformation U, the matrix labelling merely changing the 'dash', the two solutions have been obtained merely by interchanging the matrices between the radix and the unit positions. This is the process of 'reversal'. This property of reversibility can apply only to solutions in Categories One and Two, the constituent matrices of which are all from Categories One and Two and labelled with roman capitals, the digits in their principal diagonals always adding to six.

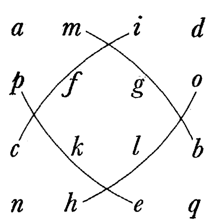
Since, in Category Three, (7, 5) matrices can combine only with (2, 10) matrices, the (7, 5) matrices always in the radix position and the (2, 10) matrices always in the unit position, the labelling in Category Three always shows an arabic numeral followed by a lower-case letter and these solutions are *not* reversible as the matrices' diagonals do not have elements adding to six.

To identify any randomly-found magic square given in the normal 1 to 16 notation with those in the symmetrical list of solutions given here in the 00–33 scale 4 notation, subtract 1 from each number in the 1–16 solution and write the square in 00–33 notation. Take note of the diagonals and determine to which category of solutions it belongs. Match the rows, columns and Frénicle quads to a lead solution (or identify the pattern) and identify the solution within the appropriate block. Familiarity with the magic squares makes recognition instant whatever the orientation.

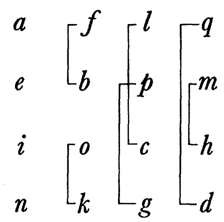
The interchanges by which solutions within blocks (and between blocks of pandiagonal and semi-pandiagonal solutions) can be obtained from one another are mostly very simple: interchanges between whole rows and/or columns or between even numbers of pairs of numbers in easily-seen patterns. The symbols '*c.*', '*r.*' show when solutions can be obtained from others by using complements or reversals respectively, but these transformations are not unique. When an interlinking is not immediately obvious a 'new' solution is labelled 'Lead'. We have not thought it necessary to show the details of links between these Lead Solutions – that they can be obtained from one another by an even number of interchanges is already established. There are however five particularly pretty sets of interchanges which characterize links between solutions which have Frénicle quadsets of the same genus Π and of the same genus Θ, and these are illustrated below by using the standard Frénicle magic square notation

a	*b*	*c*	*d*
e	*f*	*g*	*h*
i	*k*	*l*	*m*
n	*o*	*p*	*q*

Solution 9 is derived from Solution 1 by

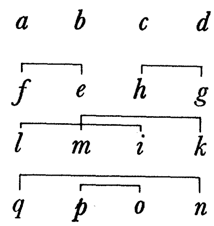


Solution 17 is derived from Solution 1 by



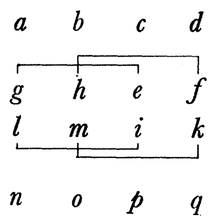
which interchanges rows with the principal and short diagonals.

Solution 33 which is the reversal of Solution 17 is also obtained from 17 by



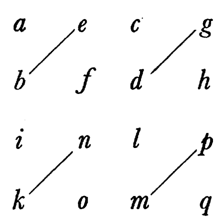
which interchanges columns with the principal and short diagonals.

Solution 25 is derived from Solution 17 by



which interchanges the centre and horizontal Frénicle quads.

These sets of interchanges are repeated for all six blocks of solutions with Frénicle quadsets of genus Π. The second and third of the interchanges illustrated above give transformations for Solutions 297 from 289 and 313 from 297 and so on in the solutions with Frénicle quadsets of genus Θ. The other interchanges within the blocks of solutions with Frénicle quadsets of genus Θ are looked after by those associated with the pattern (3) as described earlier, in addition to the easily-seen interchanges between whole rows and/or columns and the interchanges by which Solution 449 can be derived from Solution 369, namely



Appendix VI. List of all matrices

Category One

Diagonals (6 × 6): (0 1 2 3 × 0 1 2 3)

U S US

A. (2) (6)

(1)	0	1	3	2
	3	2	0	1
(6)	2	3	1	0
	1	0	2	3

A′ (1) (6)

(2)	0	3	1	2
	2	1	3	0
(6)	3	0	2	1
	1	2	0	3

E. (2) (4) (6)

(1)	0	3	1	2
(4)	3	0	2	1
(6)	2	1	3	0
	1	2	0	3

E′ (1) (5) (6)

(2)	0	1	3	2
(5)	2	3	1	0
(6)	3	2	0	1
	1	0	2	3

H. (1) (2) (6)

(3)	0	3	3	0
	1	2	2	1
(6)	2	1	1	2
	3	0	0	3

H′ (1) (2) (6)

(3)	0	3	3	0
	2	1	1	2
(6)	1	2	2	1
	3	0	0	3

I. (1) (2) (6)

(3)	1	2	2	1
	0	3	3	0
(6)	3	0	0	3
	2	1	1	2

I′ (1) (2) (6)

(3)	1	2	2	1
	3	0	0	3
(6)	0	3	3	0
	2	1	1	2

B. (3) (4)

(2)	0	1	2	3
	2	3	0	1
(4)	3	2	1	0
	1	0	3	2

B′ U (3) (5)

(1)	0	2	1	3
	3	1	2	0
(5)	2	0	3	1
	1	3	0	2

C. (3) (4)

(2)	0	2	1	3
	1	3	0	2
(4)	3	1	2	0
	2	0	3	1

C′ U (3) (5)

(1)	0	1	2	3
	3	2	1	0
(5)	1	0	3	2
	2	3	0	1

P. (3) (9)

0	0	3	3
2	3	0	1
3	2	1	0
1	1	2	2

P′ (3) (7)

0	3	0	3
3	1	2	0
2	0	3	1
1	2	1	2

R. (3) (9)

0	0	3	3
1	3	0	2
3	1	2	0
2	2	1	1

R′ (3) (7)

0	3	0	3
3	2	1	0
1	0	3	2
2	1	2	1

Q. (3) (10)

0	1	2	3
3	3	0	0
2	2	1	1
1	0	3	2

Q′ (3) (8)

0	2	1	3
2	1	2	1
3	0	3	0
1	3	0	2

T. (3) (10)

0	2	1	3
3	3	0	0
1	1	2	2
2	0	3	1

T′ (3) (8)

0	1	2	3
1	2	1	2
3	0	3	0
2	3	0	1

F. (2) (3) (4) (9) (10)

(1)	0	0	3	3
	3	3	0	0
(4)	2	2	1	1
	1	1	2	2

F′ (1) (3) (5) (7) (8)

(2)	0	3	0	3
	2	1	2	1
(5)	3	0	3	0
	1	2	1	2

G. (2) (3) (4) (9) (10)

(1)	0	0	3	3
	3	3	0	0
(4)	1	1	2	2
	2	2	1	1

G′ (1) (3) (5) (7) (8)

(2)	0	3	0	3
	1	2	1	2
(5)	3	0	3	0
	2	1	2	1

Diagonals (6 × 6): (0 0 3 3 × 0 0 3 3) or (1 1 2 2 × 1 1 2 2)

U S US

J.

		(3)	(4)	(5)
	0	1	2	3
(3)	2	3	0	1
(4)	1	0	3	2
(5)	3	2	1	0

M.

			(3)	
(3)	0	2	1	3
	2	3	0	1
	1	0	3	2
	3	1	2	0

M′

			(3)	
(3)	0	1	2	3
	1	3	0	2
	2	0	3	1
	3	2	1	0

K.

		(3)	(4)	(5)
	1	0	3	2
(3)	3	2	1	0
(4)	0	1	2	3
(5)	2	3	0	1

N.

			(3)	
(3)	1	3	0	2
	3	2	1	0
	0	1	2	3
	2	0	3	1

N′

			(3)	
(3)	1	0	3	2
	0	2	1	3
	3	1	2	0
	2	3	0	1

Category Two

Diagonals (6 × 6); (0 2 2 2 × 1 1 1 3)

U S US

V.

			(3)	
(7)	0	0	3	3
	3	2	1	0
	2	1	2	1
	1	3	0	2

V′

			(3)	
(9)	0	3	0	3
	2	2	1	1
	3	1	2	0
	1	0	3	2

W.

			(3)	
(8)	2	3	0	1
	0	0	3	3
	3	1	2	0
	1	2	1	2

W′

			(3)	
(10)	2	0	3	1
	3	2	1	0
	0	3	0	3
	1	1	2	2

X.

			(3)	
(7)	0	3	0	3
	3	2	1	0
	2	1	2	1
	1	0	3	2

X′

			(3)	
(9)	0	0	3	3
	2	2	1	1
	3	1	2	0
	1	3	0	2

Y.

			(3)	
(8)	2	3	0	1
	3	0	3	0
	0	1	2	3
	1	2	1	2

Y′

			(3)	
(10)	2	0	3	1
	0	2	1	3
	3	3	0	0
	1	1	2	2

Category Three

Diagonals (7 × 5): (0 2 2 3 × 0 1 1 3)

U S S

1. (1) (3) (12)

0	0	3	3
3	3	0	0
2	1	2	1
1	2	1	2

1′ (2) (3) (11)

0	3	0	3
2	2	1	1
3	0	3	0
1	1	2	2

2. (1) (3) (12)

0	0	3	3
3	3	0	0
1	2	1	2
2	1	2	1

2′ (2) (3) (11)

0	3	0	3
1	1	2	2
3	0	3	0
2	2	1	1

3. (3)

0	2	1	3
3	3	0	0
2	1	2	1
1	0	3	2

3′ (3)

0	1	2	3
2	2	1	1
3	0	3	0
1	3	0	2

4. (3)

0	0	3	3
1	3	0	2
3	2	1	0
2	1	2	1

4′ (3)

0	3	0	3
3	1	2	0
1	0	3	2
2	2	1	1

5. (3)

0	0	3	3
2	3	0	1
3	1	2	0
1	2	1	2

5′ (3)

0	3	0	3
3	2	1	0
2	0	3	1
1	1	2	2

6. (3)

0	1	2	3
3	3	0	0
1	2	1	2
2	0	3	1

6′ (3)

0	2	1	3
1	1	2	2
3	0	3	0
2	3	0	1

7. (2) (3)

0	2	1	3
2	3	0	1
3	1	2	0
1	0	3	2

7′ (1) (3)

0	1	2	3
3	2	1	0
2	0	3	1
1	3	0	2

8. (2) (3)

0	1	2	3
1	3	0	2
3	2	1	0
2	0	3	1

8′ (1) (3)

0	2	1	3
3	1	2	0
1	0	3	2
2	3	0	1

Diagonals (7 × 5): (0 1 3 3 × 0 0 2 3)

9. (3) (2) (7) (8)

0	3	3	0
1	1	2	2
2	2	1	1
3	0	0	3

9′ (3) (1) (7) (8)

0	3	3	0
2	1	2	1
1	2	1	2
3	0	0	3

10. (3) (2) (9) (10)

1	1	2	2
3	0	0	3
0	3	3	0
2	2	1	1

10′ (3) (1) (9) (10)

1	2	1	2
0	3	3	0
3	0	0	3
2	1	2	1

Diagonals (7 × 5): (0 1 3 3 × 0 1 2 2) or (1 1 2 3 × 0 0 2 3)

11. (11) (7)

0	3	1	2
3	1	2	0
2	0	3	1
1	2	0	3

11′ (12) (9)

0	1	3	2
2	3	0	1
3	2	1	0
1	0	2	3

12. $(11)^{-1}$ (8)

1	3	0	2
3	0	2	1
2	1	3	0
0	2	1	3

12′ $(12)^{-1}$ (10)

1	0	3	2
2	3	1	0
3	2	0	1
0	1	2	3

13. comp. (11) (7)

3	0	2	1
0	2	1	3
1	3	0	2
2	1	3	0

13′ (12) (9)

3	2	0	1
1	0	3	2
0	1	2	3
2	3	1	0

14. $(11)^{-1}$ (8)

2	0	3	1
0	3	1	2
1	2	0	3
3	1	2	0

14′ $(12)^{-1}$ (10)

2	3	0	1
1	0	2	3
0	1	3	2
3	2	1	0

15. (1) (7)

0	3	1	2
2	1	2	1
3	0	3	0
1	2	0	3

15′ (2) (9)

0	1	3	2
3	3	0	0
2	2	1	1
1	0	2	3

16. (1) (8)

1	2	1	2
3	0	2	1
2	1	3	0
0	3	0	3

16′ (2) (10)

1	1	2	2
2	3	1	0
3	2	0	1
0	0	3	3

17. comp. (1) (7)

3	0	2	1
1	2	1	2
0	3	0	3
2	1	3	0

17′ (2) (9)

3	2	0	1
0	0	3	3
1	1	2	2
2	3	1	0

18. (1) (8)

2	1	2	1
0	3	1	2
1	2	0	3
3	0	3	0

18′ (2) (10)

2	2	1	1
1	0	2	3
0	1	3	2
3	3	0	0

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Diagonals (7 × 5): (0 1 3 3 × 0 0 2 3) or (1 1 2 3 × 0 1 2 2)

U, S, US

19. (7) (3)

0	1	2	3
3	3	0	0
1	0	3	2
2	2	1	1

19′ (9) (3) — U

0	2	1	3
1	3	0	2
3	0	3	0
2	1	2	1

20. (8) (3) — S

0	0	3	3
2	3	0	1
1	1	2	2
3	2	1	0

20′ (10) (3) — US

0	3	0	3
1	2	1	2
2	0	3	1
3	1	2	0

21. (7) (3)

0	2	1	3
3	3	0	0
1	0	3	2
2	1	2	1

21′ (9) (3)

0	1	2	3
1	3	0	2
3	0	3	0
2	2	1	1

22. (8) (3)

0	0	3	3
1	3	0	2
2	1	2	1
3	2	1	0

22′ (10) (3)

0	3	0	3
2	2	1	1
1	0	3	2
3	1	2	0

23. (8) (3)

1	0	3	2
3	3	0	0
0	1	2	3
2	2	1	1

23′ (10) (3)

1	3	0	2
0	2	1	3
3	0	3	0
2	1	2	1

24. (7) (3)

0	0	3	3
3	2	1	0
1	1	2	2
2	3	0	1

24′ (9) (3)

0	3	0	3
1	2	1	2
3	1	2	0
2	0	3	1

25. (10) (3)

1	2	1	2
3	3	0	0
0	1	2	3
2	0	3	1

25′ (8) (3)

1	1	2	2
0	2	1	3
3	0	3	0
2	3	0	1

26. (7) (3)

0	0	3	3
1	2	1	2
3	1	2	0
2	3	0	1

26′ (7) (3)

0	3	0	3
3	2	1	0
1	1	2	2
2	0	3	1

Diagonals (2 × 10): (0 0 0 2 × 1 3 3 3)

a. (10) (3) $(12)^{-1}$

0	2	1	3
2	0	3	1
1	1	2	2
3	3	0	0

a′ (8) (3) $(11)^{-1}$

0	1	2	3
1	2	1	2
2	3	0	1
3	0	3	0

b. (9) (3) $(12)^{-1}$

0	2	1	3
2	0	3	1
3	3	0	0
1	1	2	2

b′ (7) (3) $(11)^{-1}$

0	1	2	3
3	0	3	0
2	3	0	1
1	2	1	2

c. (8) (3)

0	3	0	3
2	0	3	1
1	1	2	2
3	2	1	0

c′ (10) (3)

0	0	3	3
1	2	1	2
2	3	0	1
3	1	2	0

d. (7) (3)

0	2	1	3
3	0	3	0
2	3	0	1
1	1	2	2

d′ (9) (3)

0	1	2	3
2	0	3	1
3	3	0	0
1	2	1	2

Diagonals (2 × 10): (0 0 1 1 × 2 2 3 3)

U S US

e′ (3) (1) (3) (7) (9)

0	3	0	3
2	1	2	1
1	2	1	2
3	0	3	0

e. (3) (2) (3) (3) (10)

0	0	3	3
1	1	2	2
2	2	1	1
3	3	0	0

f′ (3) (1) (3) (7) (9)

1	2	1	2
3	0	3	0
0	3	0	3
2	1	2	1

f. (3) (2) (3) (8) (10)

1	1	2	2
0	0	3	3
3	3	0	0
2	2	1	1

g. (3) (3) (8) (9)

0	0	3	3
2	1	2	1
1	2	1	2
3	3	0	0

g′ (3) (3) (7) (10)

0	3	0	3
1	1	2	2
2	2	1	1
3	0	3	0

h. (3) (3) (7) (10)

1	2	1	2
0	0	3	3
3	3	0	0
2	1	2	1

h′ (3) (3) (8) (9)

1	1	2	2
3	0	3	0
0	3	0	3
2	2	1	1

j. (1) (11) $(11)^{-1}$ (1) (3) (7) (9)

0	3	0	3
3	0	3	0
1	2	1	2
2	1	2	1

j′ (2) (12) $(12)^{-1}$ (2) (3) (8) (10)

0	0	3	3
1	1	2	2
3	3	0	0
2	2	1	1

k. (3) (9)

0	1	2	3
3	0	3	0
1	2	1	2
2	3	0	1

k′ (3) (10)

0	2	1	3
1	1	2	2
3	3	0	0
2	0	3	1

m. (3) (7)

0	3	0	3
[1	0	3	2]
[3	2	1	0]
2	1	2	1

m′ (3) (8)

0	0	3	3
3	1	2	0
1	3	0	2
2	2	1	1

n. (2) (3)

0	1	2	3
1	0	3	2
3	2	1	0
2	3	0	1

n′ (1) (3)

0	2	1	3
3	1	2	0
1	3	0	2
2	0	3	1

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Appendix VII. List of all solutions

Category One

						U						S						US	
1.					2.					3.					4.				
B.B	00	12	23	31	B′B′	00	23	12	31	B.B	33	21	10	02	B′B′	33	10	21	02
Lead	21	33	02	10	*s*–*r*	32	11	20	03	*s*–*r*	12	00	31	23	*s*–*r*	01	22	13	30
s–*r*	32	20	11	03		21	02	33	10		01	13	22	30		12	31	00	23
(4)	13	01	30	22	(5)	13	30	01	22	(4)	20	32	03	11	(5)	20	03	32	11
5.					6.					7.					8.				
B.B	23	31	00	12	B′B′	23	00	31	12	B.B	10	02	33	21	B′B′	10	33	02	21
f 1.	02	10	21	33		11	32	03	20	*r* 5.	31	23	12	00	*r* 6.	22	01	30	13
	11	03	32	20		02	21	10	33		22	30	01	13		31	12	23	00
(4)	30	22	13	01	(5)	30	13	22	01	(4)	03	11	20	32	(5)	03	20	11	32
9.					10.					11.					12.				
F.F	00	03	32	31	F′F′	00	32	03	31	F.F	33	30	01	02	F′F′	33	01	30	02
f 1.	30	33	02	01	*s*–*r*	23	11	20	12	*s*–*r*	03	00	31	32	*s*–*r*	10	22	13	21
s–*r*	23	20	11	12		30	02	33	01		10	13	22	21		03	31	00	32
(4)	13	10	21	22	(5)	13	21	10	22	(4)	20	23	12	11	(5)	20	12	23	11
13.					14.					15.					16.				
F.F	32	31	00	03	F′F′	32	00	31	03	F.F	01	02	33	30	F′F′	01	33	02	30
f 9.	02	01	30	33		11	23	12	20	r 13.	31	32	03	00	*r* 14.	22	10	21	13
	11	12	23	20		02	30	01	33		22	21	10	13		31	03	32	00
(4)	21	22	13	10	(5)	21	13	22	10	(4)	12	11	20	23	(5)	12	20	11	23

17.					18.					19.					20.				
A′C′	00	33	11	22	A.C	00	11	33	22	A′C′	12	21	03	30	A.C	12	03	21	30
f 1.	21	12	30	03		32	23	01	10		33	00	22	11		20	31	13	02
	32	01	23	10		21	30	12	03		20	13	31	02		33	22	00	11
(1)	13	20	02	31	(2)	13	02	20	31	(1)	01	32	10	23	(2)	01	10	32	23
21.					22.					23.					24.				
A′C′	33	00	22	11	A.C	33	22	00	11	A′C′	21	12	30	03	A.C	21	30	12	03
f 17	12	21	03	30		01	10	32	23		00	33	11	22		13	02	20	31
	01	32	10	23		12	03	21	30		13	20	02	31		00	11	33	22
(1)	20	13	31	02	(2)	20	31	13	02	(1)	32	01	23	10	(2)	32	23	01	10
25.					26.					27.					28.				
E′G	00	33	11	22	E.G′	00	11	33	22	E′G	03	30	12	21	E.G′	03	12	30	21
f 17	30	03	21	12		23	32	10	01		33	00	22	11		20	31	13	02
	23	10	32	01		30	21	03	12		20	13	31	02		33	22	00	11
(1)	13	20	02	31	(2)	13	02	20	31	(1)	10	23	01	32	(2)	10	01	23	32
29.					30.					31.					32.				
E′G	33	00	22	11	E.G′	33	22	00	11	E′G	30	03	21	12	E.G′	30	21	03	12
f 25	03	30	12	21		10	01	23	32		00	33	11	22		13	02	20	31
	10	23	01	32		03	12	30	21		13	20	02	31		00	11	33	22
(1)	20	13	31	02	(2)	20	31	13	02	(1)	23	10	32	01	(2)	23	32	10	01

33.					34.					35.					36.				
C′A′	00	33	11	22	C.A	00	11	33	22	C′A′	21	12	30	03	C.A	21	30	12	03
r 17.	12	21	03	30		23	32	10	01		33	00	22	11		02	13	31	20
	23	10	32	01		12	03	21	30		02	31	13	20		33	22	00	11
(1)	31	02	20	13	(2)	31	20	02	13	(1)	10	23	01	32	(2)	10	01	23	32
37.					38.					39.					40.				
C′A′	33	00	22	11	C.A	33	22	00	11	C′A′	12	21	03	30	C.A	12	03	21	30
r 21.	21	12	30	03		10	01	23	32		00	33	11	22		31	20	02	13
	10	23	01	32		21	30	12	03		31	02	20	13		00	11	33	22
(1)	02	31	13	20	(2)	02	13	31	20	(1)	23	10	32	01	(2)	23	32	10	01
41.					42.					43.					44.				
G.E′	00	33	11	22	G′E	00	11	33	22	G.E′	30	03	21	12	G′E	30	21	03	12
r 25.	03	30	12	21		32	23	01	10		33	00	22	11		02	13	31	20
	32	01	23	10		03	12	30	21		02	31	13	20		33	22	00	11
(1)	31	02	20	13	(2)	31	20	02	13	(1)	01	32	10	23	(2)	01	10	32	23
45.					46.					47.					48.				
G.E′	33	00	22	11	G′E	33	22	00	11	G.E′	03	30	12	21	G′E	03	12	30	21
r 29.	30	03	21	12		01	10	32	23		00	33	11	22		31	20	02	13
	01	32	10	23		30	21	03	12		31	02	20	13		00	11	33	22
(1)	02	31	13	20	(2)	02	13	31	20	(1)	32	01	23	10	(2)	32	23	01	10

VII. 1. Π_1 (includes reversals)

Category One

								U					S					US	
49.					50.					51.					52.				
C.C	00	12	31	23	C′C′	00	31	12	23	C.C	33	21	02	10	C′C′	33	02	21	10
f 1	21	33	10	02	*s–r*	13	22	01	30	*s–r*	12	00	23	31	*s–r*	20	11	32	03
s–r	13	01	22	30		21	10	33	02		20	32	11	03		12	23	00	31
(4)	32	20	03	11	(5)	32	03	20	11	(4)	01	13	30	22	(5)	01	30	13	22
53.					54.					55.					56.				
C.C	31	23	00	12	C′C′	31	00	23	12	C.C	02	10	33	21	C′C′	02	33	10	21
f 49.	10	02	21	33		22	13	30	01	*r* 53.	23	31	12	00	*r* 54.	11	20	03	32
	22	30	13	01		10	21	02	33		11	03	20	32		23	12	31	00
(4)	03	11	32	20	(5)	03	32	11	20	(4)	30	22	01	13	(5)	30	01	22	13
57.					58.					59.					60.				
G.G	00	30	13	23	G′G′	00	13	30	23	G.G	33	03	20	10	G′G′	33	20	03	10
f 49.	03	33	10	20	*s–r*	31	22	01	12	*s–r*	30	00	23	13	*s–r*	02	11	32	21
s–r	31	01	22	12		03	10	33	20		02	32	11	21		30	23	00	13
(4)	32	02	21	11	(5)	32	21	02	11	(4)	01	31	12	22	(5)	01	12	31	22
61.					62.					63.					64.				
G.G	13	23	00	30	G′G′	13	00	23	30	G.G	20	10	33	03	G′G′	20	33	10	03
f 57.	10	20	03	33		22	31	12	01	*r* 61.	23	13	30	00	*r* 62.	11	02	21	32
	22	12	31	01		10	03	20	33		11	21	02	32		23	30	13	00
(4)	21	11	32	02	(5)	21	32	11	02	(4)	12	22	01	31	(5)	12	01	22	31

65.					66.					67.					68.				
B′A	00	33	22	11	B.A′	00	22	33	11	B′A	12	21	30	03	B.A′	12	30	21	03
f 49.	21	12	03	30		13	31	20	02		33	00	11	22		01	23	32	10
	13	20	31	02		21	03	12	30		01	32	23	10		33	11	00	22
(1)	32	01	10	23	(2)	32	10	01	23	(1)	20	13	02	31	(2)	20	02	13	31
69.					70.					71.					72.				
B′A	33	00	11	22	B.A′	33	11	00	22	B′A	21	12	03	30	B.A′	21	03	12	30
f 65.	12	21	30	03		20	02	13	31		00	33	22	11		32	10	01	23
	20	13	02	31		12	30	21	03		32	01	10	23		00	22	33	11
(1)	01	32	23	10	(2)	01	23	32	10	(1)	13	20	31	02	(2)	13	31	20	02
73.					74.					75.					76.				
F.E′	00	33	22	11	F′E	00	22	33	11	F.E′	30	03	12	21	F′E	30	12	03	21
f 65.	03	30	21	12		31	13	02	20		33	00	11	22		01	23	32	10
	31	02	13	20		03	21	30	12		01	32	23	10		33	11	00	22
(1)	32	01	10	23	(2)	32	10	01	23	(1)	02	31	20	13	(2)	02	20	31	13
77.					78.					79.					80.				
F.E′	33	00	11	22	F′E	33	11	00	22	F.E′	03	30	21	12	F′E	03	21	30	12
f 73.	30	03	12	21		02	20	31	13		00	33	22	11		32	10	01	23
	02	31	20	13		30	12	03	21		32	01	10	23		00	22	33	11
(1)	01	32	23	10	(2)	01	23	32	10	(1)	31	02	13	20	(2)	31	13	02	20

81.					82.					83.					84.				
A.B′	00	33	22	11	A′B	00	22	33	11	A.B′	21	12	03	30	A′B	21	03	12	30
r 65.	12	21	30	03		31	13	02	20		33	00	11	22		10	32	23	01
	31	02	13	20		12	30	21	03		10	23	32	01		33	11	00	22
(1)	23	10	01	32	(2)	23	01	10	32	(1)	02	31	20	13	(2)	02	20	31	13
85.					86.					87.					88.				
A.B′	33	00	11	22	A′B	33	11	00	22	A.B′	12	21	30	03	A′B	12	30	21	03
r 69.	21	12	03	30		02	20	31	13		00	33	22	11		23	01	10	32
	02	31	20	13		21	03	12	30		23	10	01	32		00	22	33	11
(1)	10	23	32	01	(2)	10	32	23	01	(1)	31	02	13	20	(2)	31	13	02	20
89.					90.					91.					92.				
E′F	00	33	22	11	E.F′	00	22	33	11	E′F	03	30	21	12	E.F′	03	21	30	12
r 73.	30	03	12	21		13	31	20	02		33	00	11	22		10	32	23	01
	13	20	31	02		30	12	03	21		10	23	32	01		33	11	00	22
(1)	23	10	01	32	(2)	23	01	10	32	(1)	20	13	02	31	(2)	20	02	13	31
93.					94.					95.					96.				
E′F	33	00	11	22	E.F′	33	11	00	22	E′F	30	03	12	21	E.F′	30	12	03	21
r 77.	03	30	21	12		20	02	13	31		00	33	22	11		23	01	10	32
	20	13	02	31		03	21	30	12		23	10	01	32		00	22	33	11
(1)	10	23	32	01	(2)	10	32	23	01	(1)	13	20	31	02	(2)	13	31	20	02

VII. 2. Π_2 (includes reversals)

Category One

							U					S					US		
97.					98.					99.					100.				
B.C	00	11	23	32	B′C′	00	23	11	32	B.C	33	22	10	01	B′C′	33	10	22	01
f 1.	22	33	01	10		31	12	20	03		11	00	32	23		02	21	13	30
	31	20	12	03		22	01	33	10		02	13	21	30		11	32	00	23
(4)	13	02	30	21	(5)	13	30	02	21	(4)	20	31	03	12	(5)	20	03	31	12
101.					102.					103.					104.				
B.C	23	32	00	11	B′C′	23	00	32	11	B.C	10	01	33	22	B′C′	10	33	01	22
f 97.	01	10	22	33		12	31	03	20		32	23	11	00		21	02	30	13
	12	03	31	20		01	22	10	33		21	30	02	13		32	11	23	00
(4)	30	21	13	02	(5)	30	13	21	02	(4)	03	12	20	31	(5)	03	20	12	31
105.					106.					107.					108.				
F.G	00	03	31	32	F′G′	00	31	03	32	F.G	33	30	02	01	F′G′	33	02	30	01
f 97.	30	33	01	02		23	12	20	11		03	00	32	31		10	21	13	22
	23	20	12	11		30	01	33	02		10	13	21	22		03	32	00	31
(4)	13	10	22	21	(5)	13	22	10	21	(4)	20	23	11	12	(5)	20	11	23	12
109.					110.					111.					112.				
F.G	31	32	00	03	F′G′	31	00	32	03	F.G	02	01	33	30	F′G′	02	33	01	30
f 105.	01	02	30	33		12	23	11	20		32	31	03	00		21	10	22	13
	12	11	23	20		01	30	02	33		21	22	10	13		32	03	31	00
(4)	22	21	13	10	(5)	22	13	21	10	(4)	11	12	20	23	(5)	11	20	12	23

113.					114.					115.					116.				
A′B′	00	33	12	21	A.B	00	12	33	21	A′B′	11	22	03	30	A.B	11	03	22	30
f 97.	22	11	30	03		31	23	02	10		33	00	21	12		20	32	13	01
	31	02	23	10		22	30	11	03		20	13	32	01		33	21	00	12
(1)	13	20	01	32	(2)	13	01	20	32	(1)	02	31	10	23	(2)	02	10	31	23
117.					118.					119.					120.				
A′B′	33	00	21	12	A.B	33	21	00	12	A′B′	22	11	30	03	A.B	22	30	11	03
f 113.	11	22	03	30		02	10	31	23		00	33	12	21		13	01	20	32
	02	31	10	23		11	03	22	30		13	20	01	32		00	12	33	21
(1)	20	13	32	01	(2)	20	32	13	01	(1)	31	02	23	10	(2)	31	23	02	10
121.					122.					123.					124.				
E′F	00	33	12	21	E.F′	00	12	33	21	E′F	03	30	11	22	E.F′	03	11	30	22
f 113.	30	03	22	11		23	31	10	02		33	00	21	12		20	32	13	01
	23	10	31	02		30	22	03	11		20	13	32	01		33	21	00	12
(1)	13	20	01	32	(2)	13	01	20	32	(1)	10	23	02	31	(2)	10	02	23	31
125.					126.					127.					128.				
E′F	33	00	21	12	E.F′	33	21	00	12	E′F	30	03	22	11	E.F′	30	22	03	11
f 121.	03	30	11	22		10	02	23	31		00	33	12	21		13	01	20	32
	10	23	02	31		03	11	30	22		13	20	01	32		00	12	33	21
(1)	20	13	32	01	(2)	20	32	13	01	(1)	23	10	31	02	(2)	23	31	10	02

129.					130.					131.					132.				
C′A	00	33	12	21	C.A′	00	12	33	21	C′A	22	11	30	03	C.A′	22	30	11	03
f 113.	11	22	03	30		23	31	10	02		33	00	21	12		01	13	32	20
	23	10	31	02		11	03	22	30		01	32	13	20		33	21	00	12
(1)	32	01	20	13	(2)	32	20	01	13	(1)	10	23	02	31	(2)	10	02	23	31
133.					134.					135.					136.				
C′A	33	00	21	12	C.A′	33	21	00	12	C′A	11	22	03	30	C.A′	11	03	22	30
f 129.	22	11	30	03		10	02	23	31		00	33	12	21		32	20	01	13
	10	23	02	31		22	30	11	03		32	01	20	13		00	12	33	21
(1)	01	32	13	20	(2)	01	13	32	20	(1)	23	10	31	02	(2)	23	31	10	02
137.					138.					139.					140.				
G.E′	00	33	12	21	G′E	00	12	33	21	G.E′	30	03	22	11	G′E	30	22	03	11
f 129.	03	30	11	22		31	23	02	10		33	00	21	12		01	13	32	20
	31	02	23	10		03	11	30	22		01	32	13	20		33	21	00	12
(1)	32	01	20	13	(2)	32	20	01	13	(1)	02	31	10	23	(2)	02	10	31	23
141.					142.					143.					144.				
G.E′	33	00	21	12	G′E	33	21	00	12	G.E′	03	30	11	22	G′E	03	11	30	22
f 137.	30	03	22	11		02	10	31	23		00	33	12	21		32	20	01	13
	02	31	10	23		30	22	03	11		32	01	20	13		00	12	33	21
(1)	01	32	13	20	(2)	01	13	32	20	(1)	31	02	23	10	(2)	31	23	02	10

VII. 3. Π_3 (reversals opposite)

Category One

						U					S					US			
145.					146.					147.					148.				
C.B	00	11	32	23	C′B′	00	32	11	23	C.B	33	22	01	10	C′B′	33	01	22	10
r 97.	22	33	10	01		13	21	02	30		11	00	23	32		20	12	31	03
	13	02	21	30		22	10	33	01		20	31	12	03		11	23	00	32
(4)	31	20	03	12	(5)	31	03	20	12	(4)	02	13	30	21	(5)	02	30	13	21
149.					150.					151.					152.				
C.B	32	23	00	11	C′B′	32	00	23	11	C.B	01	10	33	22	C′B′	01	33	10	22
r 101.	10	01	22	33		21	13	30	02		23	32	11	00		12	20	03	31
	21	30	13	02		10	22	01	33		12	03	20	31		23	11	32	00
(4)	03	12	31	20	(5)	03	31	12	20	(4)	30	21	02	13	(5)	30	02	21	13
153.					154.					155.					156.				
G.F	00	30	13	23	G′F′	00	13	30	23	G.F	33	03	20	10	G′F′	33	20	03	10
r 105.	03	33	10	20		32	21	02	11		30	00	23	13		01	12	31	22
	32	02	21	11		03	10	33	20		01	31	12	22		30	23	00	13
(4)	31	01	22	12	(5)	31	22	01	12	(4)	02	32	11	21	(5)	02	11	32	21
157.					158.					159.					160.				
G.F	13	23	00	30	G′F′	13	00	23	30	G.F	20	10	33	03	G′F′	20	33	10	03
r 109.	10	20	03	33		21	32	11	02		23	13	30	00		12	01	22	31
	21	11	32	02		10	03	20	33		12	22	01	31		23	30	13	00
(4)	22	12	31	01	(5)	22	31	12	01	(4)	11	21	02	32	(5)	11	02	21	32

161.					162.					163.					164.				
B′A′	00	33	21	12	B.A	00	21	33	12	B′A′	11	22	30	03	B.A	11	30	22	03
r 113.	22	11	03	30		13	32	20	01		33	00	12	21		02	23	31	10
	13	20	32	01		22	03	11	30		02	31	23	10		33	12	00	21
(1)	31	02	10	23	(2)	31	10	02	23	(1)	20	13	01	32	(2)	20	01	13	32
165.					166.					167.					168.				
B′A′	33	00	12	21	B.A	33	12	00	21	B′A′	22	11	03	30	B.A	22	03	11	30
r 117.	11	22	30	03		20	01	13	32		00	33	21	12		31	10	02	23
	20	13	01	32		11	30	22	03		31	02	10	23		00	21	33	12
(1)	02	31	23	10	(2)	02	23	31	10	(1)	13	20	32	01	(2)	13	32	20	01
169.					170.					171.					172.				
F.E′	00	33	21	12	F′E	00	21	33	12	F.E′	30	03	11	22	F′E	30	11	03	22
r 121.	03	30	22	11		32	13	01	20		33	00	12	21		02	23	31	10
	32	01	13	20		03	22	30	11		02	31	23	10		33	12	00	21
(1)	31	02	10	23	(2)	31	10	02	23	(1)	01	32	20	13	(2)	01	20	32	13
173.					174.					175.					176.				
F.E′	33	00	12	21	F′E	33	12	00	21	F.E′	03	30	22	11	F′E	03	22	30	11
r 125.	30	03	11	22		01	20	32	13		00	33	21	12		31	10	02	23
	01	32	20	13		30	11	03	22		31	02	10	23		00	21	33	12
(1)	02	31	23	10	(2)	02	23	31	10	(1)	32	01	13	20	(2)	32	13	01	20

177.					178.					179.					180.				
A.C′	00	33	21	12	A′C	00	21	33	12	A.C′	22	11	03	30	A′C	22	03	11	30
r 129.	11	22	30	03		32	13	01	20		33	00	12	21		10	31	23	02
	32	01	13	20		11	30	22	03		10	23	31	02		33	12	00	21
(1)	23	10	02	31	(2)	23	02	10	31	(1)	01	32	20	13	(2)	01	20	32	13
181.					182.					183.					184.				
A.C′	33	00	12	21	A′C	33	12	00	21	A.C′	11	22	30	03	A′C	11	30	22	03
r 123.	22	11	03	30		01	20	32	13		00	33	21	12		23	02	10	31
	01	32	20	13		22	03	11	30		23	10	02	31		00	21	33	12
(1)	10	23	31	02	(2)	10	31	23	02	(1)	32	01	13	20	(2)	32	13	01	20
185.					186.					187.					188.				
E′G	00	33	21	12	E.G′	00	21	33	12	E′G	03	30	22	11	E.G′	03	22	30	11
r 137.	30	03	11	22		13	32	20	01		33	00	12	21		10	31	23	02
	13	20	32	01		30	11	03	22		10	23	31	02		33	12	00	21
(1)	23	10	02	31	(2)	23	02	10	31	(1)	20	13	01	32	(2)	20	01	13	32
189.					190.					191.					192.				
E′G	33	00	12	21	E.G′	33	12	00	21	E′G	30	03	11	22	E.G′	30	11	03	22
r 141.	03	30	22	11		20	01	13	32		00	33	21	12		23	02	10	31
	20	13	01	32		03	22	30	11		23	10	02	31		00	21	33	12
(1)	10	23	31	02	(2)	10	31	23	02	(1)	13	20	32	01	(2)	13	32	20	01

VII. 4. Π_4 (reversals opposite)

Category One

							U					S					US		
193.					194.					195.					196.				
E′J	00	11	32	23	E.J	00	32	11	23	E′J	33	22	01	10	E.J	33	01	22	10
f 97.	22	33	10	01		31	03	20	12		11	00	23	32		02	30	13	21
	31	20	03	12		22	10	33	01		02	13	30	21		11	23	00	32
(4)	13	02	21	30	(5)	13	21	02	30	(4)	20	31	12	03	(5)	20	12	31	03

197.					198.					199.					200.				
E′K	32	23	00	11	E.K	32	00	23	11	E′K	01	10	33	22	E.K	01	33	10	22
f 193.	10	01	22	33		03	31	12	20		23	32	11	00		30	02	21	13
	03	12	31	20		10	22	01	33		30	21	02	13		23	11	32	00
(4)	21	30	13	02	(5)	21	13	30	02	(4)	12	03	20	31	(5)	12	20	03	31

201.					202.					203.					204.				
E′J	00	12	31	23	E.J	00	31	12	23	E′J	33	21	02	10	E.J	33	02	21	10
f 193.	21	33	10	02		32	03	20	11		12	00	23	31		01	30	13	22
	32	20	03	11		21	10	33	02		01	13	30	22		12	23	00	31
(4)	13	01	22	30	(5)	13	22	01	30	(4)	20	32	11	03	(5)	20	11	32	03

205.					206.					207.					208.				
E′J	31	23	00	12	E.J	31	00	23	12	E′J	02	10	33	21	E.J	02	33	10	21
f 201.	10	02	21	33		03	32	11	20		23	31	12	00		30	01	22	13
	03	11	32	20		10	21	02	33		30	22	01	13		23	12	31	00
(4)	22	30	13	01	(5)	22	13	30	01	(4)	11	03	20	32	(5)	11	20	03	32

209.					210.					211.					212.				
F′H	00	33	03	30	F.H′	00	03	33	30	F′I	11	22	12	21	F.I′	11	12	22	21
f 193.	22	11	21	12		31	32	02	01		33	00	30	03		20	23	13	10
	31	02	32	01		22	21	11	12		20	13	23	10		33	30	00	03
(1)	13	20	10	23	(2)	13	10	20	23	(1)	02	31	01	32	(2)	02	01	31	32

213.					214.					215.					216.				
F′H	33	00	30	03	F.H′	33	30	00	03	F′I	22	11	21	12	F.I′	22	21	11	12
f 209.	11	22	12	21		02	01	31	32		00	33	03	30		13	10	20	23
	02	31	01	32		11	12	22	21		13	20	10	23		00	03	33	30
(1)	20	13	23	10	(2)	20	23	13	10	(1)	31	02	32	01	(2)	31	32	02	01

217.					218.					219.					220.				
F′H′	00	33	03	30	F.H	00	03	33	30	F′I′	12	21	11	22	F.I	12	11	21	22
f 209.	21	12	22	11		32	31	01	02		33	00	30	03		20	23	13	10
	32	01	31	02		21	22	12	11		20	13	23	10		33	30	00	03
(1)	13	20	10	23	(2)	13	10	20	23	(1)	01	32	02	31	(2)	01	02	32	31

221.					222.					223.					224.				
F′H′	33	00	30	03	F.H	33	30	00	03	F′I′	21	12	22	11	F.I	21	22	12	11
f 217.	12	21	11	22		01	02	32	31		00	33	03	30		13	10	20	23
	01	32	02	31		12	11	21	22		13	20	10	23		00	03	33	30
(1)	20	13	23	10	(2)	20	23	13	10	(1)	32	01	31	02	(2)	32	31	01	02

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225.					226.					227.					228.				
G′H′	00	33	03	30	G.H	00	03	33	30	G′I′	22	11	21	12	G.I	22	21	11	12
f 209.	11	22	12	21		32	31	01	02		33	00	30	03		10	13	23	20
	32	01	31	02		11	12	22	21		10	23	13	20		33	30	00	03
(1)	23	10	20	13	(2)	23	20	10	13	(1)	01	32	02	31	(2)	01	02	32	31

229.					230.					231.					232.				
G′H′	33	00	30	03	G.H	33	30	00	03	G′I′	11	22	12	21	G.I	11	12	22	21
f 225.	22	11	21	12		01	02	32	31		00	33	03	30		23	20	10	13
	01	32	02	31		22	21	11	12		23	10	20	13		00	03	33	30
(1)	10	23	13	20	(2)	10	13	23	20	(1)	32	01	31	02	(2)	32	31	01	02

233.					234.					235.					236.				
G′H	00	33	03	30	G.H′	00	03	33	30	G′I	21	12	22	11	G.I′	21	22	12	11
f 225.	12	21	11	22		31	32	02	01		33	00	30	03		10	13	23	20
	31	02	32	01		12	11	21	22		10	23	13	20		33	30	00	03
(1)	23	10	20	13	(2)	23	20	10	13	(1)	02	31	01	32	(2)	02	01	31	32

237.					238.					239.					240.				
G′H	33	00	30	03	G.H′	33	30	00	03	G′I	12	21	11	22	G.I′	12	11	21	22
f 233.	21	12	22	11		02	01	31	32		00	33	03	30		23	20	10	13
	02	31	01	32		21	22	12	11		23	10	20	13		00	03	33	30
(1)	10	23	13	20	(2)	10	13	23	20	(1)	31	02	32	01	(2)	31	32	02	01

VII. 5. Π_5 (reversals opposite)

Category One

							U					S					US		
241.					242.					243.					244.				
J.E′	00	11	23	32	J.E	00	23	11	32	J.E′	33	22	10	01	J.E	33	10	22	01
r 193.	22	33	01	10		13	30	02	21		11	00	32	23		20	03	31	12
	13	02	30	21		22	01	33	10		20	31	03	12		11	32	00	23
(4)	31	20	12	03	(5)	31	12	20	03	(4)	02	13	21	30	(5)	02	21	13	30
245.					246.					247.					248.				
K.E′	23	32	00	11	K.E	23	00	32	11	K.E′	10	01	33	22	K.E	10	33	01	22
r 197.	01	10	22	33		30	13	21	02		32	23	11	00		03	20	12	31
	30	21	13	02		01	22	10	33		03	12	20	31		32	11	23	00
(4)	12	03	31	20	(5)	12	31	03	20	(4)	21	30	02	13	(5)	21	02	30	13
249.					250.					251.					252.				
J.E′	00	21	13	32	J.E	00	13	21	32	J.E′	33	12	20	01	J.E	33	20	12	01
r 201.	12	33	01	20		23	30	02	11		21	00	32	13		10	03	31	22
	23	02	30	11		12	01	33	20		10	31	03	22		21	32	00	13
(4)	31	10	22	03	(5)	31	22	10	03	(4)	02	23	11	30	(5)	02	11	23	30
253.					254.					255.					256.				
K.E′	13	32	00	21	K.E	13	00	32	21	K.E′	20	01	33	12	K.E	20	33	01	12
r 205.	01	20	12	33		30	23	11	02		32	13	21	00		03	10	22	31
	30	11	23	02		01	12	20	33		03	22	10	31		32	21	13	00
(4)	22	03	31	10	(5)	22	31	03	10	(4)	11	30	02	23	(5)	11	02	30	23

257.					258.					259.					260.				
H.F′	00	33	30	03	H′F	00	30	33	03	I.F′	11	22	21	12	I′F	11	21	22	12
r 209.	22	11	12	21		13	23	20	10		33	00	03	30		02	32	31	01
	13	20	23	10		22	12	11	21		02	31	32	01		33	03	00	30
(1)	31	02	01	32	(2)	31	01	02	32	(1)	20	13	10	23	(2)	20	10	13	23
261.					262.					263.					264.				
H.F′	33	00	03	30	H′F	33	03	00	30	I.F′	22	11	12	21	I′F	22	12	11	21
r 213.	11	22	21	12		20	10	13	23		00	33	30	03		31	01	02	32
	20	13	10	23		11	21	22	12		31	02	01	32		00	30	33	03
(1)	02	31	32	01	(2)	02	32	31	01	(1)	13	20	23	10	(2)	13	23	20	10
265.					266.					267.					268.				
H′F′	00	33	30	03	H.F	00	30	33	03	I′F′	21	12	11	22	I.F	21	11	12	22
r 217.	12	21	22	11		23	13	10	20		33	00	03	30		02	32	31	01
	23	10	13	20		12	22	21	11		02	31	32	01		33	03	00	30
(1)	31	02	01	32	(2)	31	01	02	32	(1)	10	23	20	13	(2)	10	20	23	13
269.					270.					271.					272.				
H′F′	33	00	03	30	H.F.	33	03	00	30	I′F′	12	21	22	11	I.F	12	22	21	11
r 221.	21	12	11	22		10	20	23	13		00	33	30	03		31	01	02	32
	10	23	20	13		21	11	12	22		31	02	01	32		00	30	33	03
(1)	02	31	32	01	(2)	02	32	31	01	(1)	23	10	13	20	(2)	23	13	10	20

273.					274.					275.					276.				
H′G′	00	33	30	03	H.G	00	30	33	03	I′G′	22	11	12	21	I.G	22	12	11	21
r 225.	11	22	21	12		23	13	10	20		33	00	03	30		01	31	32	02
	23	10	13	20		11	21	22	12		01	32	31	02		33	03	00	30
(1)	32	01	02	31	(2)	32	02	01	31	(1)	10	23	20	13	(2)	10	20	23	13
277.					278.					279.					280.				
H′G′	33	00	03	30	H.G	33	03	00	30	I′G′	11	22	21	12	I.G	11	21	22	12
r 229.	22	11	12	21		10	20	23	13		00	33	30	03		32	02	01	31
	10	23	20	13		22	12	11	21		32	01	02	31		00	30	33	03
(1)	01	32	31	02	(2)	01	31	32	02	(1)	23	10	13	20	(2)	23	13	10	20
281.					282.					283.					284.				
H.G′	00	33	30	03	H′G	00	30	33	03	I.G′	12	21	22	11	I′G	12	22	21	11
r 233.	21	12	11	22		13	23	20	10		33	00	03	30		01	31	32	02
	13	20	23	10		21	11	12	22		01	32	31	02		33	03	00	30
(1)	32	01	02	31	(2)	32	02	01	31	(1)	20	13	10	23	(2)	20	10	13	23
285.					286.					287.					288.				
H.G′	33	00	03	30	H′G	33	03	00	30	I.G′	21	12	11	22	I′G	21	11	12	22
r 237.	12	21	22	11		20	10	13	23		00	33	30	03		32	02	01	31
	20	13	10	23		12	22	21	11		32	01	02	31		00	30	33	03
(1)	01	32	31	02	(2)	01	31	32	02	(1)	13	20	23	10	(2)	13	23	20	10

VII. 6. Π_6 (reversals opposite)

Category One

					U					S					US				
289.					290.					291.					292.				
A.A′	00	13	31	22	A′A	00	31	13	22	A.A′	21	32	10	03	A′A	21	10	32	03
f 18.	32	21	03	10		23	12	30	01		13	00	22	31		02	33	11	20
	23	30	12	01		32	03	21	10		02	11	33	20		13	22	00	31
(6)	11	02	20	33	(6)	11	20	02	33	(6)	30	23	01	12	(6)	30	01	23	12
293.					294.					295.					296.				
E′E	00	13	31	22	E.E′	00	31	13	22	E′E	30	23	01	12	E.E′	30	01	23	12
f 289.	23	30	12	01		32	03	21	10		13	00	22	31		02	33	11	20
	32	21	03	10		23	12	30	01		02	11	33	20		13	22	00	31
(6)	11	02	20	33	(6)	11	20	02	33	(6)	21	32	10	03	(6)	21	10	32	03
297.					298.					299.					300.				
B′B	00	21	12	33	B.B′	00	12	21	33	B′B	13	32	01	20	B.B′	13	01	32	20
f 289.	32	13	20	01		23	31	02	10		21	00	33	12		30	22	11	03
	23	02	31	10		32	20	13	01		30	11	22	03		21	33	00	12
(3)	11	30	03	22	(3)	11	03	30	22	(3)	02	23	10	31	(3)	02	10	23	31
301.					302.					303.					304.				
P′P	00	30	03	33	P.P′	00	03	30	33	Q′Q	13	32	01	20	Q.Q′	13	01	32	20
f 297.	32	13	20	01		23	31	02	10		30	00	33	03		21	22	11	12
	23	02	31	10		32	20	13	01		21	11	22	12		30	33	00	03
(3)	11	21	12	22	(3)	11	12	21	22	(3)	02	23	10	31	(3)	02	10	23	31
305.					306.					307.					308.				
Q′Q	00	21	12	33	Q.Q′	00	12	21	33	P′P	13	23	10	20	P.P′	13	10	23	20
f 297.	23	13	20	10		32	31	02	01		21	00	33	12		30	22	11	03
	32	02	31	01		23	20	13	10		30	11	22	03		21	33	00	12
(3)	11	30	03	22	(3)	11	03	30	22	(3)	02	32	01	31	(3)	02	01	32	31
309.					310.					311.					312.				
F′F	00	30	03	33	F.F′	00	03	30	33	F′F	13	23	10	20	F.F′	13	10	23	20
f 297.	23	13	20	10		32	31	02	01		30	00	33	03		21	22	11	12
	32	02	31	01		23	20	13	10		21	11	22	12		30	33	00	03
(3)	11	21	12	22	(3)	11	12	21	22	(3)	02	32	01	31	(3)	02	01	32	31
313.					314.					315.					316.				
C.C′	00	21	12	33	C′C	00	12	21	33	C.C′	32	13	20	01	C′C	32	20	13	01
f 297.	13	32	01	20		31	23	10	02		21	00	33	12		03	11	22	30
	31	10	23	02		13	01	32	20		03	22	11	30		21	33	00	12
(3)	22	03	30	11	(3)	22	30	03	11	(3)	10	31	02	23	(3)	10	02	31	23
317.					318.					319.					320.				
R.R′	00	03	30	33	R′R	00	30	03	33	T.T′	32	13	20	01	T′T	32	20	13	01
f 313.	13	32	01	20		31	23	10	03		03	00	33	30		21	11	22	12
	31	10	23	02		13	01	32	20		21	22	11	12		03	33	00	30
(3)	22	21	12	11	(3)	22	12	21	11	(3)	10	31	02	23	(3)	10	02	31	23
321.					322.					323.					324.				
T′T	00	21	12	33	T.T′	00	12	21	33	R′R	32	31	02	01	R.R′	32	02	31	01
f 313.	31	32	01	02		13	23	10	20		21	00	33	12		03	11	22	30
	13	10	23	20		31	01	32	02		03	22	11	30		21	33	00	12
(3)	22	03	30	11	(3)	22	30	03	11	(3)	10	13	20	23	(3)	10	20	13	23
325.					326.					327.					328.				
G.G′	00	03	30	33	G′G	00	30	03	33	G.G′	32	31	02	01	G′G	32	02	31	01
f 313.	31	32	01	02		13	23	10	20		03	00	33	30		21	11	22	12
	13	10	23	20		31	01	32	02		21	22	11	12		03	33	00	30
(3)	22	21	12	11	(3)	22	12	21	11	(3)	10	13	20	23	(3)	10	20	13	23

VII. 7. Θ_1 (includes reversals caused by U)

Category One

						U					S					US			
329.					330.					331.					332.				
A.A′	13	00	22	31	A′A	13	22	00	31	A.A′	32	21	03	10	A′A	32	03	21	10
f 289.	21	32	10	03		30	01	23	12		00	13	31	22		11	20	02	33
	30	23	01	12		21	10	32	03		11	02	20	33		00	31	13	22
(6)	02	11	33	20	(6)	02	33	11	20	(6)	23	30	12	01	(6)	23	12	30	01
333.					334.					335.					336.				
E′E	13	00	22	31	E.E′	13	22	00	31	E′E	23	30	12	01	E.E′	23	12	30	01
f 329.	30	23	01	12		21	10	32	03		00	13	31	22		11	20	02	33
	21	32	10	03		30	01	23	12		11	02	20	33		00	31	13	22
(6)	02	11	33	20	(6)	02	33	11	20	(6)	32	21	03	10	(6)	32	03	21	10
337.					338.					339.					340.				
B′B	21	00	33	12	B.B′	21	33	00	12	B′B	32	13	20	01	B.B′	32	20	13	01
f 297.	13	32	01	20		02	10	23	31		00	21	12	33		11	03	30	22
	02	23	10	31		13	01	32	20		11	30	03	22		00	12	21	33
(3)	30	11	22	03	(3)	30	22	11	03	(3)	23	02	31	10	(3)	23	31	02	10
341.					342.					343.					344.				
P′P	21	11	22	12	P.P′	21	22	11	12	Q′Q	32	13	20	01	Q.Q′	32	20	13	01
f 337.	13	32	01	20		02	10	23	31		11	21	12	22		00	03	30	33
	02	23	10	31		13	01	32	20		00	30	03	33		11	12	21	22
(3)	30	00	33	03	(3)	30	33	00	03	(3)	23	02	31	10	(3)	23	31	02	10
345.					346.					347.					348.				
Q′Q	21	00	33	12	Q.Q′	21	33	00	12	P′P	32	02	31	01	P.P′	32	31	02	01
f 337.	02	32	01	31		13	10	23	20		00	21	12	33		11	03	30	22
	13	23	10	20		02	01	32	31		11	30	03	22		00	12	21	33
(3)	30	11	22	03	(3)	30	22	11	03	(3)	23	13	20	10	(3)	23	20	13	10
349.					350.					351.					352.				
F′F	21	11	22	12	F.F′	21	22	11	12	F′F	32	02	31	01	F.F′	32	31	02	01
f 337.	02	32	01	31		13	10	23	20		11	21	12	22		00	03	30	33
	13	23	10	20		02	01	32	31		00	30	03	33		11	12	21	22
(3)	30	00	33	03	(3)	30	33	00	03	(3)	23	13	20	10	(3)	23	20	13	10
353.					354.					355.					356.				
C.C′	21	00	33	12	C′C	21	33	00	12	C.C′	13	32	01	20	C′C	13	01	32	20
f 313.	32	13	20	01		10	02	31	23		00	21	12	33		22	30	03	11
	10	31	02	23		32	20	13	01		22	03	30	11		00	12	21	33
(3)	03	22	11	30	(3)	03	11	22	30	(3)	31	10	23	02	(3)	31	23	10	02
357.					358.					359.					360.				
R.R′	21	22	11	12	R′R	21	11	22	12	T.T′	13	32	01	20	T′T	13	01	32	20
f 353.	32	13	20	01		10	02	31	23		22	21	12	11		00	30	03	33
	10	31	02	23		32	20	13	01		00	03	30	33		22	12	21	11
(3)	03	00	33	30	(3)	03	33	00	30	(3)	31	10	23	02	(3)	31	23	10	02
361.					362.					363.					364.				
T.T′	21	00	33	12	T′T	21	33	00	12	R.R′	13	10	23	20	R′R	13	23	10	20
f 353.	10	13	20	23		32	02	31	01		00	21	12	33		22	30	03	11
	32	31	02	01		10	20	13	23		22	03	30	11		00	12	21	33
(3)	03	22	11	30	(3)	03	11	22	30	(3)	31	32	01	02	(3)	31	01	32	02
365.					366.					367.					368.				
G.G′	21	22	11	12	G′G	21	11	22	12	G.G′	13	10	23	20	G′G	13	23	10	20
f 353.	10	13	20	23		32	02	31	01		22	21	12	11		00	30	03	33
	32	31	02	01		10	20	13	23		00	03	30	33		22	12	21	11
(3)	03	00	33	30	(3)	03	33	00	30	(3)	31	32	01	02	(3)	31	01	32	02

VII. 8. Θ_1 (includes reversals caused by U)

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Category One

							U					S					US		
369.					370.					371.					372.				
A.A	00	13	32	21	A′A′	00	32	13	21	A.A	22	31	10	03	A′A′	22	10	31	03
f 289.	31	22	03	10	*s*–*r*	23	11	30	02	*s*–*r*	13	00	21	32	*s*–*r*	01	33	12	20
s–*r*	23	30	11	02		31	03	22	10		01	12	33	20		13	21	00	32
(6)	12	01	20	33	(6)	12	20	01	33	(6)	30	23	02	11	(6)	30	02	23	11

373.					374.					375.					376.				
E′E	00	13	32	21	E.E′	00	32	13	21	E′E	30	23	02	11	E.E′	30	02	23	11
f 369.	23	30	11	02	*r* 373.	31	03	22	10		13	00	21	32	*r* 375.	01	33	12	20
	31	22	03	10		23	11	30	02		01	12	33	20		13	21	00	32
(6)	12	01	20	33	(6)	12	20	01	33	(6)	22	31	10	03	(6)	22	10	31	03

377.					378.					379.					380.				
B′C′	00	22	11	33	B.C	00	11	22	33	B′C′	13	31	02	20	B.C	13	02	31	20
f 369.	31	13	20	02		23	32	01	10		22	00	33	11		30	21	12	03
	23	01	32	10		31	20	13	02		30	12	21	03		22	33	00	11
(3)	12	30	03	21	(3)	12	03	30	21	(3)	01	23	10	32	(3)	01	10	23	32

381.					382.					383.					384.				
P′R	00	30	03	33	P.R′	00	03	30	33	Q′T	13	31	02	20	Q.T′	13	02	31	20
f 377.	31	13	20	02		23	32	01	10		30	00	33	03		22	21	12	11
	23	01	32	10		31	20	13	02		22	12	21	11		30	33	00	03
(3)	12	22	11	21	(3)	12	11	22	21	(3)	01	23	10	32	(3)	01	10	23	32

385.					386.					387.					388.				
Q′T	00	22	11	33	Q.T′	00	11	22	33	P′R	13	23	10	20	P.R′	13	10	23	20
f 377.	23	13	20	10		31	32	01	02		22	00	33	11		30	21	12	03
	31	01	32	02		23	20	13	10		30	12	21	03		22	33	00	11
(3)	12	30	03	21	(3)	12	03	30	21	(3)	01	31	02	32	(3)	01	02	31	32

389.					390.					391.					392.				
F′G	00	30	03	33	F.G′	00	03	30	33	F′G	13	23	10	20	F.G′	13	10	23	20
f 377.	23	13	20	10		31	32	01	02		30	00	33	03		22	21	12	11
	31	01	32	02		23	20	13	10		22	12	21	11		30	33	00	03
(3)	12	22	11	21	(3)	12	11	22	21	(3)	01	31	02	32	(3)	01	02	31	32

393.					394.					395.					396.				
C′B′	00	22	11	33	C.B	00	11	22	33	C′B′	31	13	20	02	C.B	31	20	13	02
r 377.	13	31	02	20		32	23	10	01		22	00	33	11		03	12	21	30
	32	10	23	01		13	02	31	20		03	21	12	30		22	33	00	11
(3)	21	03	30	12	(3)	21	30	03	12	(3)	10	32	01	23	(3)	10	01	32	23

397.					398.					399.					400.				
R.P′	00	03	30	33	R′P	00	30	03	33	T.Q′	31	13	20	02	T′Q	31	20	13	02
r 381.	13	31	02	20		32	23	10	01		03	00	33	30		22	12	21	11
	32	10	23	01		13	02	31	20		22	21	12	11		03	33	00	30
(3)	21	22	11	12	(3)	21	11	22	12	(3)	10	32	01	23	(3)	10	01	32	23

401.					402.					403.					404.				
T.Q′	00	22	11	33	T′Q	00	11	22	33	R.P′	31	32	01	02	R′P	31	01	32	02
r 385.	32	31	02	01		13	23	10	20		22	00	33	11		03	12	21	30
	13	10	23	20		32	02	31	01		03	21	12	30		22	33	00	11
(3)	21	03	30	12	(3)	21	30	03	12	(3)	10	13	20	23	(3)	10	20	13	23

405.					406.					407.					408.				
G.F′	00	03	30	33	G′F	00	30	03	33	G.F′	31	32	01	02	G′F	31	01	32	02
r 389.	32	31	02	01		13	23	10	20		03	00	33	30		22	12	21	11
	13	10	23	20		32	02	31	01		22	21	12	11		03	33	00	30
(3)	21	22	11	12	(3)	21	11	22	12	(3)	10	13	20	23	(3)	10	20	13	23

VII. 9. Θ_2 (includes reversals)

Category One

						U						S						US	
409.					410.					411.					412.				
A.A	13	00	21	32	A′A′	13	21	00	32	A.A	31	22	03	10	A′A′	31	03	22	10
f 369.	22	31	10	03		30	02	23	11	*r* 409.	00	13	32	21	*r* 410.	12	20	01	33
	30	23	02	11		22	10	31	03		12	01	20	33		00	32	13	21
(6)	01	12	33	20	(6)	01	33	12	20	(6)	23	30	11	02	(6)	23	11	30	02
413.					414.					415.					416.				
E′E	13	00	21	32	E.E′	13	21	00	32	E′E	23	30	11	02	E.E′	23	11	30	02
f 409.	30	23	02	11		22	10	31	03	*r* 413.	00	13	32	21	*r* 414.	12	20	01	33
	22	31	10	03		30	02	23	11		12	01	20	33		00	32	13	21
(6)	01	12	33	20	(6)	01	33	12	20	(6)	31	22	03	10	(6)	31	03	22	10

417.					418.					419.					420.				
B′C′	22	00	33	11	B.C	22	33	00	11	B′C′	31	13	20	02	B.C	31	20	13	02
f 377.	13	31	02	20		01	10	23	32		00	22	11	33		12	03	30	21
	01	23	10	32		13	02	31	20		12	30	03	21		00	11	22	33
(3)	30	12	21	03	(3)	30	21	12	03	(3)	23	01	32	10	(3)	23	32	01	10
421.					422.					423.					424.				
P′R	22	12	21	11	P.R′	22	21	12	11	Q′T	31	13	20	02	Q.T′	31	20	13	02
f 417.	13	31	02	20		01	10	23	32		12	22	11	21		00	03	30	33
	01	23	10	32		13	02	31	20		00	30	03	33		12	11	22	21
(3)	30	00	33	03	(3)	30	33	00	03	(3)	23	01	32	10	(3)	23	32	01	10
425.					426.					427.					428.				
Q′T	22	00	33	11	Q.T′	22	33	00	11	P′R	31	01	32	02	P.R′	31	32	01	02
f 417.	01	31	02	32		13	10	23	20		00	22	11	33		12	03	30	21
	13	23	10	20		01	02	31	32		12	30	03	21		00	11	22	33
(3)	30	12	21	03	(3)	30	21	12	03	(3)	23	13	20	10	(3)	23	20	13	10
429.					430.					431.					432.				
F′G	22	12	21	11	F.G′	22	21	12	11	F′G	31	01	32	02	F.G′	31	32	01	02
f 417.	01	31	02	32		13	10	23	20		12	22	11	21		00	03	30	33
	13	23	10	20		01	02	31	32		00	30	03	33		12	11	22	21
(3)	30	00	33	03	(3)	30	33	00	03	(3)	23	13	20	10	(3)	23	20	13	10

433.					434.					435.					436.				
C′B′	22	00	33	11	C.B	22	33	00	11	C′B′	13	31	02	20	C.B	13	02	31	20
r 417.	31	13	20	02		10	01	32	23		00	22	11	33		21	30	03	12
	10	32	01	23		31	20	13	02		21	03	30	12		00	11	22	33
(3)	03	21	12	30	(3)	03	12	21	30	(3)	32	10	23	01	(3)	32	23	10	01
437.					438.					439.					440.				
R.P′	22	21	12	11	R′P	22	12	21	11	T.Q′	13	31	02	20	T′Q	13	02	31	20
r 421.	31	13	20	02		10	01	32	23		21	22	11	12		00	30	03	33
	10	32	01	23		31	20	13	02		00	03	30	33		21	11	22	12
(3)	03	00	33	30	(3)	03	33	00	30	(3)	32	10	23	01	(3)	32	23	10	01
441.					442.					443.					444.				
T.Q′	22	00	33	11	T′Q	22	33	00	11	R.P′	13	10	23	20	R′P	13	23	10	20
r 425.	10	13	20	23		31	01	32	02		00	22	11	33		21	30	03	12
	31	32	01	02		10	20	13	23		21	03	30	12		00	11	22	33
(3)	03	21	12	30	(3)	03	12	21	30	(3)	32	31	02	01	(3)	32	02	31	01
445.					446.					447.					448.				
G.F′	22	21	12	11	G′F	22	12	21	11	G.F′	13	10	23	20	G′F	13	23	10	20
r 429.	10	13	20	23		31	01	32	02		21	22	11	12		00	30	03	33
	31	32	01	02		10	20	13	23		00	03	30	33		21	11	22	12
(3)	03	00	33	30	(3)	03	33	00	30	(3)	32	31	02	01	(3)	32	02	31	01

VII. 10. Θ_2 (includes reversals)

Category One

					U					S					US				
449.					450.					451.					452.				
H′H′	00	31	32	03	H.H	00	32	31	03	I′I′	22	13	10	21	I.I	22	10	13	2
f 369.	13	22	21	10	*s–r*	23	11	12	20	*s–r*	31	00	03	32	*s–r*	01	33	30	0
s–r	23	12	11	20		13	21	22	10		01	30	33	02		31	03	00	3
(6)	30	01	02	33	(6)	30	02	01	33	(6)	12	23	20	11	(6)	12	20	23	1
453.					454.					455.					456.				
H.H′	00	31	32	03	H′H	00	32	31	03	I.I′	12	23	20	11	I′I	12	20	23	1
f 449.	23	12	11	20	*r* 453.	13	21	22	10		31	00	03	32	*r* 455.	01	33	30	0
	13	22	21	10		23	11	12	20		01	30	33	02		31	03	00	3
(6)	30	01	02	33	(6)	30	02	01	33	(6)	22	13	10	21	(6)	22	10	13	2

457.					458.					459.					460.				
J.H	00	22	11	33	J.H′	00	11	22	33	J.I	31	13	20	02	J.I′	31	20	13	0
f 449.	13	31	02	20		23	32	01	10		22	00	33	11		12	03	30	2
	23	01	32	10		13	02	31	20		12	30	03	21		22	33	00	1
(3)	30	12	21	03	(3)	30	21	12	03	(3)	01	23	10	32	(3)	01	10	23	3
461.					462.					463.					464.				
M.H	00	12	21	33	M′H′	00	21	12	33	M′I	31	13	20	02	M.I′	31	20	13	0
f 457.	13	31	02	20		23	32	01	10		12	00	33	21		22	03	30	1
	23	01	32	10		13	02	31	20		22	30	03	11		12	33	00	2
(3)	30	22	11	03	(3)	30	11	22	03	(3)	01	23	10	32	(3)	01	10	23	3
465.					466.					467.					468.				
M′H	00	22	11	33	M.H′	00	11	22	33	M.I	31	23	10	02	M′I′	31	10	23	0
f 457.	23	31	02	10		13	32	01	20		22	00	33	11		12	03	30	2
	13	01	32	20		23	02	31	10		12	30	03	21		22	33	00	1
(3)	30	12	21	03	(3)	30	21	12	03	(3)	01	13	20	32	(3)	01	20	13	3
469.					470.					471.					472.				
J.H	00	12	21	33	J.H′	00	21	12	33	J.I	31	23	10	02	J.I′	31	10	23	0
f 457.	23	31	02	10		13	32	01	20		12	00	33	21		22	03	30	11
	13	01	32	20		23	02	31	10		22	30	03	11		12	33	00	21
(3)	30	22	11	03	(3)	30	11	22	03	(3)	01	13	20	32	(3)	01	20	13	3

473.					474.					475.					476.				
H.J	00	22	11	33	H.J	00	11	22	33	I.J.	13	31	02	20	I′J	13	02	31	2
r 457.	31	13	20	02		32	23	10	01		22	00	33	11		21	30	03	1
	32	10	23	01		31	20	13	02		21	03	30	12		22	33	00	11
(3)	03	21	12	30	(3)	03	12	21	30	(3)	10	32	01	23	(3)	10	01	32	2
477.					478.					479.					480.				
H.M	00	21	12	33	H′M′	00	12	21	33	I.M′	13	31	02	20	I′M	13	02	31	2
r 461.	31	13	20	02		32	23	10	01		21	00	33	12		22	30	03	11
	32	10	23	01		31	20	13	02		22	03	30	11		21	33	00	12
(3)	03	22	11	30	(3)	03	11	22	30	(3)	10	32	01	23	(3)	10	01	32	23
481.					482.					483.					484.				
H.M′	00	22	11	33	H′M	00	11	22	33	I.M	13	32	01	20	I′M′	13	01	32	20
r 465.	32	13	20	01		31	23	10	02		22	00	33	11		21	30	03	12
	31	10	23	02		32	20	13	01		21	03	30	12		22	33	00	11
(3)	03	21	12	30	(3)	03	12	21	30	(3)	10	31	02	23	(3)	10	02	31	23
485.					486.					487.					488.				
H.J	00	21	12	33	H′J	00	12	21	33	I.J	13	32	01	20	I′J	13	01	32	20
r 469.	32	13	20	01		31	23	10	02		21	00	33	12		22	30	03	11
	31	10	23	02		32	20	13	01		22	03	30	11		21	33	00	12
(3)	03	22	11	30	(3)	03	11	22	30	(3)	10	31	02	23	(3)	10	02	31	23

VII. 11. Θ_3 (includes reversals)

Category One

						U					S					US			
489.					490.					491.					492.				
H′I′	31	00	03	32	H.I	31	03	00	32	H′I′	13	22	21	10	H.I	13	21	22	10
f 449.	22	13	10	21		12	20	23	11	*r* 489.	00	31	32	03	*r* 490.	30	02	01	33
	12	23	20	11		22	10	13	21		30	01	02	33		00	32	31	03
(6)	01	30	33	02	(6)	01	33	30	02	(6)	23	12	11	20	(6)	23	11	12	20
493.					494.					495.					496.				
H.I′	31	00	03	32	H′I	31	03	00	32	H.I′	23	12	11	20	H′I	23	11	12	20
f 489.	12	23	20	11		22	10	13	21	*r* 494.	00	31	32	03	*r* 493.	30	02	01	33
	22	13	10	21		12	20	23	11		30	01	02	33		00	32	31	03
(6)	01	30	33	02	(6)	01	33	30	02	(6)	13	22	21	10	(6)	13	21	22	10

497.					498.					499.					500.				
K.H	22	00	33	11	K.H′	22	33	00	11	K.I	13	31	02	20	K.I′	13	02	31	20
f 457.	31	13	20	02		01	10	23	32		00	22	11	33		30	21	12	03
	01	23	10	32		31	20	13	02		30	12	21	03		00	11	22	33
(3)	12	30	03	21	(3)	12	03	30	21	(3)	23	01	32	10	(3)	23	32	01	10
501.					502.					503.					504.				
N.H	22	30	03	11	N′H′	22	03	30	11	N′I	13	31	02	20	N.I′	13	02	31	20
f 497.	31	13	20	02		01	10	23	32		30	22	11	03		00	21	12	33
	01	23	10	32		31	20	13	02		00	12	21	33		30	11	22	03
(3)	12	00	33	21	(3)	12	33	00	21	(3)	23	01	32	10	(3)	23	32	01	10
505.					506.					507.					508.				
N′H	22	00	33	11	N.H′	22	33	00	11	N.I	13	01	32	20	N′I′	13	32	01	20
f 497.	01	13	20	32		31	10	23	02		00	22	11	33		30	21	12	03
	31	23	10	02		01	20	13	32		30	12	21	03		00	11	22	33
(3)	12	30	03	21	(3)	12	03	30	21	(3)	23	31	02	10	(3)	23	02	31	10
509.					510.					511.					512.				
K.H	22	30	03	11	K.H′	22	03	30	11	K.I	13	01	32	20	K.I′	13	32	01	20
f 497.	01	13	20	32		31	10	23	02		30	22	11	03		00	21	12	33
	31	23	10	02		01	20	13	32		00	12	21	33		30	11	22	03
(3)	12	00	33	21	(3)	12	33	00	21	(3)	23	31	02	10	(3)	23	02	31	10

513.					514.					515.					516.				
H.K	22	00	33	11	H′K	22	33	00	11	I.K	31	13	20	02	I′K	31	20	13	02
r 497.	13	31	02	20		10	01	32	23		00	22	11	33		03	12	21	30
	10	32	01	23		13	02	31	20		03	21	12	30		00	11	22	33
(3)	21	03	30	12	(3)	21	30	03	12	(3)	32	10	23	01	(3)	32	23	10	01
517.					518.					519.					520.				
H.N	22	03	30	11	H′N′	22	30	03	11	I.N′	31	13	20	02	I′N	31	20	13	02
r 501.	13	31	02	20		10	01	32	23		03	22	11	30		00	12	21	33
	10	32	01	23		13	02	31	20		00	21	12	33		03	11	22	30
(3)	21	00	33	12	(3)	21	33	00	12	(3)	32	10	23	01	(3)	32	23	10	01
521.					522.					523.					524.				
H.N′	22	00	33	11	H′N	22	33	00	11	I.N	31	10	23	02	I′N′	31	23	10	02
r 505.	10	31	02	23		13	01	32	20		00	22	11	33		03	12	21	30
	13	32	01	20		10	02	31	23		03	21	12	30		00	11	22	33
(3)	21	03	30	12	(3)	21	30	03	12	(3)	32	13	20	01	(3)	32	20	13	01
525.					526.					527.					528.				
H.K	22	03	30	11	H′K	22	30	03	11	I.K.	31	10	23	02	I′K	31	23	10	02
r 509.	10	31	02	23		13	01	32	20		03	22	11	30		00	12	21	33
	13	32	01	20		10	02	31	23		00	21	12	33		03	11	22	30
(3)	21	00	33	12	(3)	21	33	00	12	(3)	32	13	20	01	(3)	32	20	13	01

VII. 12. Θ_3 (includes reversals)

Category Two

							U					S					US		
529.					530.					531.					532.				
X.G′	00	33	20	13	X′G	00	20	33	13	Y.G′	22	31	02	11	Y′G	22	02	31	11
Lead	31	22	11	02		03	23	10	30		33	00	13	20		01	21	32	12
	03	10	23	30		31	11	22	02		01	32	21	12		33	13	00	20
(7)	32	01	12	21	(9)	32	12	01	21	(8)	10	03	30	23	(10)	10	30	03	23
533.					534.					535.					536.				
X.G′	33	00	13	20	X′G	33	13	00	20	T.G′	11	02	31	22	Y′G	11	31	02	22
c 529.	02	11	22	31		30	10	23	03		00	33	20	13		32	12	01	21
	30	23	10	03		02	22	11	31		32	01	12	21		00	20	33	13
(7)	01	32	21	12	(9)	01	21	32	12	(8)	23	30	03	10	(10)	23	03	30	10
537.					**538.**					**539.**					**540.**				
V.R′	00	**33**	20	**13**	V′R	00	20	**33**	**13**	W.T′	22	**03**	30	**11**	W′T	22	**30**	**03**	**11**
f 529.	03	22	**11**	30		**31**	**23**	10	02		**33**	00	**13**	20		01	21	**32**	12
	31	10	**23**	02		**03**	11	22	**30**		01	32	21	**12**		**33**	**13**	00	20
(7)	**32**	01	12	**21**	(9)	32	12	01	21	(8)	10	**31**	02	**23**	(10)	10	02	31	**23**
541.					542.					543.					544.				
V.R′	33	00	13	20	V′R	33	13	00	20	W.T′	11	30	03	22	W′T	11	03	30	22
c 537.	30	11	22	03		02	10	23	31		00	33	20	13		32	12	01	21
	02	23	10	31		30	22	11	03		32	01	12	21		00	20	33	13
(7)	01	32	21	12	(9)	01	21	32	12	(8)	23	02	31	10	(10)	23	31	02	10

545.					546.					547.					548.				
Y.G′	20	33	00	13	Y′G	20	00	33	13	X.G′	22	11	02	31	X′G	22	02	11	31
f 529.	11	22	31	02		23	03	10	30		33	20	13	00		01	21	12	32
	23	10	03	30		11	31	22	02		01	12	21	32		33	13	20	00
(8)	12	01	32	21	(10)	12	32	01	21	(7)	10	23	30	03	(9)	10	30	23	03
549.					550.					551.					552.				
Y.G′	13	00	33	20	Y′G	13	33	00	20	X.G′	11	22	31	02	X′G	11	31	22	02
c 545.	22	11	02	31		10	30	23	03		00	13	20	33		32	12	21	01
	10	23	30	03		22	02	11	31		32	21	12	01		00	20	13	33
(8)	21	32	01	12	(10)	21	01	32	12	(7)	23	10	03	30	(9)	23	03	10	30
553.					554.					555.					556.				
W.T′	20	01	32	13	W′T	20	32	01	13	V.R′	22	11	02	31	V′R	22	02	11	31
f 545.	11	22	31	02		23	03	10	30		01	20	13	32		33	21	12	00
	23	10	03	30		11	31	22	02		33	12	21	00		01	13	20	32
(8)	12	33	00	21	(10)	12	00	33	21	(7)	10	23	30	03	(9)	10	30	23	03
557.					558.					559.					560.				
W.T′	13	32	01	20	W′T	13	01	32	20	V.R′	11	22	31	02	V′R	11	31	22	02
c 553.	22	11	02	31		10	30	23	03		32	13	20	01		00	12	21	33
	10	23	30	03		22	02	11	31		00	21	12	33		32	20	13	01
(8)	21	00	33	12	(10)	21	33	00	12	(7)	23	10	03	30	(9)	23	03	10	30

VII. 13. (reversals opposite)

Solutions 529–544: X_1 Solutions 545–560: X_2

Category Two

					U					S					US				
561.					562.					563.					564.				
G′X	00	33	02	31	G.X′	00	02	33	31	G′Y	22	13	20	11	G.Y′	22	20	13	11
r 529.	13	22	11	20		30	32	01	03		33	00	31	02		10	12	23	21
	30	01	32	03		13	11	22	20		10	23	12	21		33	31	00	02
(7)	23	10	21	12	(9)	23	21	10	12	(8)	01	30	03	32	(10)	01	03	30	32
565.					566.					567.					568.				
G′X	33	00	31	02	G.X′	33	31	00	02	GY	11	20	13	22	G.Y′	11	13	20	22
c 561.	20	11	22	13		03	01	32	30		00	33	02	31		23	21	10	12
	03	32	01	30		20	22	11	13		23	10	21	12		00	02	33	31
(7)	10	23	12	21	(9)	10	12	23	21	(8)	32	03	30	01	(10)	32	30	03	01
569.					570.					571.					572.				
R′V	00	33	02	31	V′R	00	02	33	31	W.T′	22	30	03	11	W′T	22	03	30	11
r 537.	30	22	11	03		13	32	01	20		33	00	31	02		10	12	23	21
	13	01	32	20		30	11	22	03		10	23	12	21		33	31	00	02
(7)	23	10	21	12	(9)	23	21	10	12	(8)	01	13	20	32	(10)	01	20	13	32
573.					574.					575.					576.				
R′V	33	00	31	02	V′R	33	31	00	02	W.T′	11	03	30	22	W′T	11	30	03	22
c 569.	03	11	22	30		20	01	32	13		00	33	02	31		23	21	10	12
	20	32	01	13		03	22	11	30		23	10	21	12		00	02	33	31
(7)	10	23	12	21	(9)	10	12	23	21	(8)	32	20	13	01	(10)	32	13	20	01

577.					578.					579.					580.				
G′Y	02	33	00	31	G.Y′	02	00	33	31	G′X	22	11	20	13	G.X′	22	20	11	13
r 545.	11	22	13	20		32	30	01	03		33	02	31	00		10	12	21	23
	32	01	30	03		11	13	22	20		10	21	12	23		33	31	02	00
(8)	21	10	23	12	(10)	21	23	10	12	(7)	01	32	03	30	(9)	01	03	32	30
581.					582.					583.					584.				
G′Y	31	00	33	02	G.Y′	31	33	00	02	G′X	11	22	13	20	G.X′	11	13	22	20
c 577.	22	11	20	13		01	03	32	30		00	31	02	33		23	21	12	10
	01	32	03	30		22	20	11	13		23	12	21	10		00	02	31	33
(8)	12	23	10	21	(10)	12	10	23	21	(7)	32	01	30	03	(9)	32	30	01	03
585.					586.					587.					588.				
W.T′	02	10	23	31	W′T	02	23	10	31	V.R′	22	11	20	13	V′R	22	20	11	13
r 553.	11	22	13	20		32	30	01	03		10	02	31	23		33	12	21	00
	32	01	30	03		11	13	22	20		33	21	12	00		10	31	02	23
(8)	21	33	00	12	(10)	21	00	33	12	(7)	01	32	03	30	(9)	01	03	32	30
589.					590.					591.					592.				
W.T′	31	23	10	02	W′T	31	10	23	02	V.R′	11	22	13	20	V′R	11	13	22	20
c 585.	22	11	20	13		01	03	32	30		23	31	02	10		00	21	12	33
	01	32	03	30		22	20	11	13		00	12	21	33		23	02	31	10
(8)	12	00	33	21	(10)	12	33	00	21	(7)	32	01	30	03	(9)	32	30	01	03

VII. 14 (reversals opposite)

Solutions 561–567: X_3 Solutions 577–592: X_4

CATEGORY TWO

							U					S					US		
593.					594.					595.					596.				
X.F′	00	33	20	13	X′F	00	20	33	13	Y.F′	21	32	01	12	Y′F	21	01	32	12
f 529.	32	21	12	01		03	23	10	30		33	00	13	20		02	22	31	11
	03	10	23	30		32	12	21	01		02	31	22	11		33	13	00	20
(7)	31	02	11	22	(9)	31	11	02	22	(8)	10	03	30	23	(10)	10	30	03	23
597.					598.					599.					600.				
X.F′	33	00	13	20	X′F	33	13	00	20	Y.F′	12	01	32	21	Y′F	12	32	01	21
c 593.	01	12	21	32		30	10	23	03		00	33	20	13		31	11	02	22
	30	23	10	03		01	21	12	32		31	02	11	22		00	20	33	13
(7)	02	31	22	11	(9)	02	22	31	11	(8)	23	30	03	10	(10)	23	03	30	10
601.					602.					603.					604.				
V.P′	00	33	20	13	V′P	00	20	33	13	W.Q′	21	03	30	12	W′Q	21	30	03	12
f 593.	03	21	12	30		32	23	10	01		33	00	13	20		02	22	31	11
	32	10	23	01		03	12	21	30		02	31	22	11		33	13	00	20
(7)	31	02	11	22	(9)	31	11	02	22	(8)	10	32	01	23	(10)	10	01	32	23
605.					606.					607.					608.				
V.P′	33	00	13	20	V′P	33	13	00	20	W.Q′	12	30	03	21	W′Q	12	03	30	21
c 601.	30	12	21	03		01	10	23	32		00	33	20	13		31	11	02	22
	01	23	10	32		30	21	12	03		31	02	11	22		00	20	33	13
(7)	02	31	22	11	(9)	02	22	31	11	(8)	23	01	32	10	(10)	23	32	01	10
609.					610.					611.					612.				
Y.F′	20	33	00	13	Y′F	20	00	33	13	X.F′	21	12	01	32	X′F	21	01	12	32
f 593.	12	21	32	01		23	03	10	30		33	20	13	00		02	22	11	31
	23	10	03	30		12	32	21	01		02	11	22	31		33	13	20	00
(8)	11	02	31	22	(10)	11	31	02	22	(7)	10	23	30	03	(9)	10	30	23	03
613.					614.					615.					616.				
Y.F′	13	00	33	20	Y′F	13	33	00	20	X.F′	12	21	32	01	X′F	12	32	21	01
c 609.	21	12	01	32		10	30	23	03		00	13	20	33		31	11	22	02
	10	23	30	03		21	01	12	32		31	22	11	02		00	20	13	33
(8)	22	31	02	11	(10)	22	02	31	11	(7)	23	10	03	30	(9)	23	03	10	30
617.					618.					619.					620.				
W.Q′	20	02	31	13	W′Q	20	31	02	13	V.P′	21	12	01	32	V′P	21	01	12	32
f 609.	12	21	32	01		23	03	10	30		02	20	13	31		33	22	11	00
	23	10	03	30		12	32	21	01		33	11	22	00		02	13	20	31
(8)	11	33	00	22	(10)	11	00	33	22	(7)	10	23	30	03	(9)	10	30	23	03
621.					622.					623.					624.				
W.Q′	13	31	02	20	W′Q	13	02	31	20	V.P′	12	21	32	01	V′P	12	32	21	01
c 617.	21	12	01	32		10	30	23	03		31	13	20	02		00	11	22	33
	10	23	30	03		21	01	12	32		00	22	11	33		31	20	13	02
(8)	22	00	33	11	(10)	22	33	00	11	(7)	23	10	03	30	(9)	23	03	10	30

VII. 15. (reversals opposite)

Solutions 593–608: X_5 Solutions 609–624: X_6

Category Two

					U					S					US				
625. F′X	00	33	02	31	626. F.X′	00	02	33	31	627. F′Y	12	23	10	21	628. F.Y′	12	10	23	21
r 593.	23	12	21	10		30	32	01	03		33	00	31	02		20	22	13	11
	30	01	32	03		23	21	12	10		20	13	22	11		33	31	00	02
(7)	13	20	11	22	(9)	13	11	20	22	(8)	01	30	03	32	(10)	01	03	30	32

629. F′X	33	00	31	02	630. F.X′	33	31	00	02	631. F′Y	21	10	23	12	632. F.Y′	21	23	10	12
c 625.	10	21	12	23		03	01	32	30		00	33	02	31		13	11	20	22
	03	32	01	30		10	12	21	23		13	20	11	22		00	02	33	31
(7)	20	13	22	11	(9)	20	22	13	11	(8)	32	03	30	01	(10)	32	30	03	01

633. P′V	00	33	02	31	634. P.V′	00	02	33	31	635. Q′W	12	30	03	21	636. Q.W′	12	03	30	21
r 601.	30	12	21	03		23	32	01	10		33	00	31	02		20	22	13	11
	23	01	32	10		30	21	12	03		20	13	22	11		33	31	00	02
(7)	13	20	11	22	(9)	13	11	20	22	(8)	01	23	10	32	(10)	01	10	23	32

637. P′V	33	00	31	02	638. P.V′	33	31	00	02	639. Q′W	21	03	30	12	640. Q.W′	21	30	03	12
c 633.	03	21	12	30		10	01	32	23		00	33	02	31		13	11	20	22
	10	32	01	23		03	12	21	30		13	20	11	22		00	02	33	31
(7)	20	13	22	11	(9)	20	22	13	11	(8)	32	10	23	01	(10)	32	23	10	01

641. F′Y	02	33	00	31	642. F.Y′	02	00	33	31	643. F′X	12	21	10	23	644. F.X′	12	10	21	23
r 609.	21	12	23	10		32	30	01	03		33	02	31	00		20	22	11	13
	32	01	30	03		21	23	12	10		20	11	22	13		33	31	02	00
(8)	11	20	13	22	(10)	11	13	20	22	(7)	01	32	03	30	(9)	01	03	32	30

645. F′Y	31	00	33	02	646. F.Y′	31	33	00	02	647. F′X	21	12	23	10	648. F.X′	21	23	12	10
c 641.	12	21	10	23		01	03	32	30		00	31	02	33		13	11	22	20
	01	32	03	30		12	10	21	23		13	22	11	20		00	02	31	33
(8)	22	13	20	11	(10)	22	20	13	11	(7)	32	01	30	03	(9)	32	30	01	03

649. Q′W	02	20	13	31	650. Q.W′	02	13	20	31	651. P′Y	12	21	10	23	652. P.Y′	12	10	21	23
r 617.	21	12	23	10		32	30	01	03		20	02	31	13		33	22	11	00
	32	01	30	03		21	23	12	10		33	11	22	00		20	31	02	13
(8)	11	33	00	22	(10)	11	00	33	22	(7)	01	32	03	30	(9)	01	03	32	30

653. Q′W	31	13	20	02	654. Q.W′	31	20	13	02	655. P′Y	21	12	23	10	656. P.Y′	21	23	12	10
c 649.	12	21	10	23		01	03	32	30		13	31	02	20		00	11	22	33
	01	32	03	30		12	10	21	23		00	22	11	33		13	02	31	20
(8)	22	00	33	11	(10)	22	33	00	11	(7)	32	01	30	03	(9)	32	30	01	03

VII. 16. (reversals opposite)

Solutions 625–640: X_7 Solutions 641–656: X_8

Category Three

					U					S					US				
657.					658.					659.					660.				
19.*j*	00	33	10	23	19′*j*′	00	10	33	23	20.*j*	30	13	20	03	20′*j*	30	20	13	03
Lead	13	30	03	20		21	31	02	12		33	00	23	10		01	11	32	22
	21	02	31	12		13	03	30	20		01	32	11	22		33	23	00	10
(7)	32	01	22	11	(9)	32	22	01	11	(8)	02	21	12	31	(10)	02	12	21	31
661.					662.					663.					664.				
19.*j*	33	00	23	10	19′*j*′	33	23	00	10	20.*j*	03	20	13	30	20′*j*′	03	13	20	30
c 657.	20	03	30	13		12	02	31	21		00	33	10	23		32	22	01	11
	12	31	02	21		20	30	03	13		32	01	22	11		00	10	33	23
(7)	01	32	11	22	(9)	01	11	32	22	(8)	31	12	21	02	(10)	31	21	12	02
665.					666.					667.					668.				
21.*m*	00	33	10	23	21′*m*′	00	10	33	23	22.*k*	30	21	12	03	22′*k*′	30	12	21	03
f 657.	21	30	03	12		13	31	02	20		33	00	23	10		01	11	32	22
	13	02	31	20		21	03	30	12		01	32	11	22		33	23	00	10
(7)	32	01	22	11	(9)	32	22	01	11	(8)	02	13	20	31	(10)	02	20	13	31
669.					670.					671.					672.				
21.*m*	33	00	23	10	21′*m*′	33	23	00	10	22.*k*	03	12	21	30	22′*k*′	03	21	12	30
c 665.	12	03	30	21		20	02	31	13		00	33	10	23		32	22	01	11
	20	31	02	13		12	30	03	21		32	01	22	11		00	10	33	23
(7)	01	32	11	22	(9)	01	11	32	22	(8)	31	20	13	02	(10)	31	13	20	02
673.					674.					675.					676.				
23.*j*	10	33	00	23	23′*j*′	10	00	33	23	24.*j*	30	03	20	13	24′*j*′	30	20	03	13
f 657.	03	30	13	20		31	21	02	12		33	10	23	00		01	11	22	32
	31	02	21	12		03	13	30	20		01	22	11	32		33	23	10	00
(8)	22	01	32	11	(10)	22	32	01	11	(7)	02	31	12	21	(9)	02	12	31	21
677.					678.					679.					680.				
23.*j*	23	00	33	10	23′*j*′	23	33	00	10	24.*j*	03	30	13	20	24′*j*′	03	13	30	20
c 673.	30	03	20	13		02	12	31	21		00	23	10	33		32	22	11	01
	02	31	12	21		30	20	03	13		32	11	22	01		00	10	23	33
(8)	11	32	01	22	(10)	11	01	32	22	(7)	31	02	21	12	(9)	31	21	02	12
681.					682.					683.					684.				
25′*k*	10	01	32	23	25.*k*′	10	32	01	23	26′*m*	30	03	20	13	26.*m*′	30	20	03	13
f 673.	03	30	13	20		31	21	02	12		01	10	23	32		33	11	22	00
	31	02	21	12		03	13	30	20		33	22	11	00		01	23	10	32
(8)	22	33	00	11	(10)	22	00	33	11	(7)	02	31	12	21	(9)	02	12	31	21
685.					686.					687.					688.				
25′*k*	23	32	01	10	25.*k*′	23	01	32	10	26′*m*	03	30	13	20	26.*m*′	03	13	30	20
c 681.	30	03	20	13		02	12	31	21		32	23	10	01		00	22	11	33
	02	31	12	21		30	20	03	13		00	11	22	33		32	10	23	01
(8)	11	00	33	22	(10)	11	33	00	22	(7)	31	02	21	12	(9)	31	21	02	12

VII. 17. (no reversals)

Solutions 657–672: X_9 Solutions 673–688: X_{10}

Category Three

						U					S					US			
689.					690.					691.					692.				
11.*e*′	00	33	10	23	11′*e*	00	10	33	23	12.*f*′	11	32	01	22	12′*f*	11	01	32	22
Lead	32	11	22	01		21	31	02	12		33	00	23	10		20	30	13	03
	21	02	31	12		32	22	11	01		20	13	30	03		33	23	00	10
(7)	13	20	03	30	(9)	13	03	20	30	(8)	02	21	12	31	(10)	02	12	21	31
693.					694.					695.					696.				
13.*e*′	33	00	23	10	13′*e*	33	23	00	10	14.*f*′	22	01	32	11	14′*f*	22	32	01	11
c 689.	01	22	11	32		12	02	31	21		00	33	10	23		13	03	20	30
	12	31	02	21		01	11	22	32		13	20	03	30		00	10	33	23
(7)	20	13	30	03	(9)	20	30	13	03	(8)	31	12	21	02	(10)	31	21	12	02
697.					698.					699.					700.				
15.*g*′	00	33	10	23	15′*g*	00	10	33	23	16.*h*′	11	21	12	22	15′*h*	11	12	21	22
f 689.	21	11	22	12		32	31	02	01		33	00	23	10		20	30	13	03
	32	02	31	01		21	22	11	12		20	13	30	03		33	23	00	10
(7)	13	20	03	30	(9)	13	03	20	30	(8)	02	32	01	31	(10)	02	01	32	31
701.					702.					703.					704.				
17.*g*′	33	00	23	10	17′*g*	33	23	00	10	18.*h*′	22	12	21	11	18′*h*	22	21	12	11
c 697.	12	22	11	21		01	02	31	32		00	33	10	23		13	03	20	30
	01	31	02	32		12	11	22	21		13	20	03	30		00	10	33	23
(7)	20	13	30	03	(9)	20	30	13	03	(8)	31	01	32	02	(10)	31	32	01	02

705.					706.					707.					708.				
14.*e*′	10	33	00	23	14′*e*	10	00	33	23	13.*f*′	11	22	01	32	13′*f*	11	01	22	32
f 689.	22	11	32	01		31	21	02	12		33	10	23	00		20	30	03	13
	31	02	21	12		22	32	11	01		20	03	30	13		33	23	10	00
(8)	03	20	13	30	(10)	03	13	20	30	(7)	02	31	12	21	(9)	02	12	31	21
709.					710.					711.					712.				
12.*e*′	23	00	33	10	12′*e*	23	33	00	10	11.*f*′	22	11	32	01	11′*f*	22	32	11	01
c 705.	11	22	01	32		02	12	31	21		00	23	10	33		13	03	30	20
	02	31	12	21		11	01	22	32		13	30	03	20		00	10	23	33
(8)	30	13	20	03	(10)	30	20	13	03	(7)	31	02	21	12	(9)	31	21	02	12
713.					714.					715.					716.				
18.*g*	10	20	13	23	18′*g*′	10	13	20	23	17.*h*	11	22	01	32	17′*h*′	11	01	22	32
f 705.	22	11	32	01		31	21	02	12		20	10	23	13		33	30	03	00
	31	02	21	12		22	32	11	01		33	03	30	00		20	23	10	13
(8)	03	33	00	30	(10)	03	00	33	30	(7)	02	31	12	21	(9)	02	12	31	21
717.					718.					719.					720.				
16.*g*	23	13	20	10	16′*g*′	23	20	13	10	15.*h*	22	11	32	01	15′*h*′	22	32	11	01
c 713.	11	22	01	32		02	12	31	21		13	23	10	20		00	03	30	33
	02	31	12	21		11	01	22	32		00	30	03	33		13	10	23	20
(8)	30	00	33	03	(10)	30	33	00	03	(7)	31	02	21	12	(9)	31	21	02	12

VII. 18. (no reversals)

Solutions 689–704: X_{11} Solutions 705–720: X_{12}

Category Three

							U					S					US		
721.					722.					723.					724.				
11.b′	00	33	12	21	11′b	00	12	33	21	12.a′	10	31	02	23	12′a	10	02	31	23
Lead	31	10	23	02		22	30	03	11		33	00	21	12		20	32	13	01
	22	03	30	11		31	23	10	02		20	13	32	01		33	21	00	12
(7)	13	20	01	32	(9)	13	01	20	32	(8)	03	22	11	30	(10)	03	11	22	30
725.					726.					727.					728.				
13.b′	33	00	21	12	13′b	33	21	00	12	14.a′	23	02	31	10	14′a	23	31	02	10
c 721.	02	23	10	31		11	03	30	22		00	33	12	21		13	01	20	32
	11	30	03	22		02	10	23	31		13	20	01	32		00	12	33	21
(7)	20	13	32	01	(9)	20	32	13	01	(8)	30	11	22	03	(10)	30	22	11	03
729.					730.					731.					732.				
15.d	00	33	12	21	15′d	00	12	33	21	16 c	10	22	11	23	16′c′	10	11	22	23
f 721.	22	10	23	11		31	30	03	02		33	00	21	12		20	32	13	01
	31	03	30	02		22	23	10	11		20	13	32	01		33	21	00	12
(7)	13	20	01	32	(9)	13	01	20	32	(8)	03	31	02	30	(10)	03	02	31	30
733.					734.					735.					736.				
17.d	33	00	21	12	17′d′	33	21	00	12	18.c	23	11	22	10	18′c′	23	22	11	10
c 729.	11	23	10	22		02	03	30	31		00	33	12	21		13	01	20	32
	02	30	03	31		11	10	23	22		13	20	01	32		00	12	33	21
(7)	20	13	32	01	(9)	20	32	13	01	(8)	30	02	31	03	(10)	30	31	02	03

737.					738.					739.					740.				
9′j	00	33	30	03	9.j′	00	30	33	03	10.j	20	23	10	13	10′j′	20	10	23	13
Lead	23	20	13	10		11	21	12	22		33	00	03	30		01	31	32	02
	11	12	21	22		23	13	20	10		01	32	31	02		33	03	00	30
(7)	32	01	02	31	(9)	32	02	01	31	(8)	12	11	22	21	(10)	12	22	11	21
741.					742.					743.					744.				
9′j	33	00	03	30	9.j′	33	03	00	30	10.j	13	10	23	20	10′j′	13	23	10	20
c 737.	10	13	20	23		22	12	21	11		00	33	30	03		32	02	01	31
	22	21	12	11		10	20	13	23		32	01	02	31		00	30	33	03
(7)	01	32	31	02	(9)	01	31	32	02	(8)	21	22	11	12	(10)	21	11	22	12
745.					746.					747.					748.				
9.m	00	33	30	03	9′m′	00	30	33	03	10′k	20	11	22	13	10.k′	20	22	11	13
f 737.	11	20	13	22		23	21	12	10		33	00	03	30		01	31	32	02
	23	12	21	10		11	13	20	22		01	32	31	02		33	03	00	30
(7)	32	01	02	31	(9)	32	02	01	31	(8)	12	23	10	21	(10)	12	10	23	21
749.					750.					751.					752.				
9.m	33	00	03	30	9′m′	33	03	00	30	10′k	13	22	11	20	10.k′	13	11	22	20
c 745.	22	13	20	11		10	12	21	23		00	33	30	03		32	02	01	31
	10	21	12	23		22	20	13	11		32	01	02	31		00	30	33	03
(7)	01	32	31	02	(9)	01	31	32	02	(8)	21	10	23	12	(10)	21	23	10	12

VII. 19. (no reversals)

Solutions 721–736: X_{13} Solutions 737–752: X_{14}

Category Three

						U					S					US			
753.					754.					755.					756.				
1′b′	00	31	02	33	1.b	00	02	31	33	2′a′	20	23	10	13	2.a	20	10	23	13
Lead	23	20	13	10		32	30	03	01		31	00	33	02		12	22	11	21
	32	03	30	01		23	13	20	10		12	11	22	21		31	33	00	02
(3)	11	12	21	22	(3)	11	21	12	22	(3)	03	32	01	30	(3)	03	01	32	20
757.					758.					759.					760				
3′d	00	12	21	33	3.d′	00	21	12	33	4′c	20	23	10	13	4. c′	20	10	23	13
f 753.	23	20	13	10		32	30	03	01		12	00	33	21		31	22	11	02
	32	03	30	01		23	13	20	10		31	11	22	02		12	33	00	21
(3)	11	31	02	22	(3)	11	02	31	22	(3)	03	32	01	30	(3)	03	01	32	30
761.					762.					763.					764.				
5′d′	00	31	02	33	5.d	00	02	31	33	6′c′	20	32	01	13	6.c	20	01	32	13
f 753.	32	20	13	01		23	30	03	10		31	00	33	02		12	22	11	21
	23	03	30	10		32	13	20	01		12	11	22	21		31	33	00	02
(3)	11	12	21	22	(3)	11	21	12	22	(3)	03	23	10	30	(3)	03	10	23	30
765.					766.					767.					768.				
7′b	00	12	21	33	7.b′	00	21	12	33	8′a	20	32	01	13	8.a′	20	01	32	13
f 753.	32	20	13	01		23	30	03	10		12	00	33	21		31	22	11	02
	23	03	30	10		32	13	20	01		31	11	22	02		12	33	00	21
(3)	11	31	02	22	(3)	11	02	31	22	(3)	03	23	10	30	(3)	03	10	23	30

769.					770.					771.					772.				
1′a′	00	31	02	33	1.a	00	02	31	33	2′b′	22	21	12	11	2.b	22	12	21	11
Lead	21	22	11	12		32	30	03	01		31	00	33	02		10	20	13	23
	32	03	30	01		21	11	22	12		10	13	20	23		31	33	00	02
(3)	13	10	23	20	(3)	13	23	10	20	(3)	03	32	01	30	(3)	03	01	32	30
773.					774.					775.					776.				
3′c′	00	10	23	33	3.c	00	23	10	33	4′d′	22	21	12	11	4.d	22	12	21	11
f 769.	21	22	11	12		32	30	03	01		10	00	33	23		31	20	13	02
	32	03	30	01		21	11	22	12		31	13	20	02		10	33	00	23
(3)	13	31	02	20	(3)	13	02	31	20	(3)	03	32	01	30	(3)	03	01	32	30
777.					778.					779.					780.				
5′c	00	31	02	33	5.c′	00	02	31	33	6′d	22	32	01	11	6.d′	22	01	32	11
f 769.	32	22	11	01		21	30	03	12		31	00	33	02		10	20	13	23
	21	03	30	12		32	11	22	01		10	13	20	23		31	33	00	02
(3)	13	10	23	20	(3)	13	23	10	20	(3)	03	21	12	30	(3)	03	12	21	30
781.					782.					783.					784.				
7′a	00	10	23	33	7.a′	00	23	10	33	8′b	22	32	01	11	8.b′	22	01	32	11
f 769.	32	22	11	01		21	30	03	12		10	00	33	23		31	20	13	02
	21	03	30	12		32	11	22	01		31	13	20	02		10	33	00	23
(3)	13	31	02	20	(3)	13	02	31	20	(3)	03	21	12	30	(3)	03	12	21	30

785.					786.					787.					788.				
9′a′	00	21	12	33	9.a	00	12	21	33	10.b′	22	31	02	11	10′b	22	02	31	11
Lead	31	22	11	02		32	20	13	01		21	00	33	12		10	30	03	23
	32	13	20	01		31	11	22	02		10	03	30	23		21	33	00	12
(3)	03	10	23	30	(3)	03	23	10	30	(3)	13	32	01	20	(3)	13	01	32	20
789.					790.					791.					792.				
9.c′	00	10	23	33	9′c	00	23	10	33	10′d′	22	31	02	11	10.d	22	00	31	11
f 785.	31	22	11	02		32	20	13	01		10	00	33	23		21	30	03	12
	32	13	20	01		31	11	22	02		21	03	30	12		10	33	00	23
(3)	03	21	12	30	(3)	03	12	21	30	(3)	13	32	01	20	(3)	13	01	32	20
793.					794.					795.					796.				
9′c	00	21	12	33	9.c′	00	12	21	33	10.d	22	32	01	11	10′d′	22	01	32	11
f 785.	32	22	11	01		31	20	13	02		21	00	33	12		10	30	03	23
	31	13	20	02		32	11	22	01		10	03	30	23		21	33	00	12
(3)	03	10	23	30	(3)	03	23	10	30	(3)	13	31	02	20	(3)	13	02	31	20
797.					798.					799.					800.				
9.a	00	10	23	33	9′a′	00	23	10	33	10′b	22	32	01	11	10.b′	22	01	32	11
f 785.	32	22	11	01		31	20	13	02		10	00	33	23		21	30	03	12
	31	13	20	02		32	11	22	01		21	03	30	12		10	33	00	23
(3)	03	21	12	30	(3)	03	12	21	30	(3)	13	31	02	20	(3)	13	02	31	20

VII. 20. (no reversals)

Solutions 753–768: Φ_1 Solutions 769–784: Φ_2 Solutions 785–800: Φ_3

Category Three

					U					S					US				
801.					802.					803.					804.				
19.*a*′	00	13	20	33	19′*a*	00	20	13	33	20.*b*′	30	31	02	03	20.*b*′	30	02	31	03
Lead	31	30	03	02		12	32	01	21		13	00	33	20		22	10	23	11
	12	01	32	21		31	03	30	02		22	23	10	11		13	33	00	20
(3)	23	22	11	10	(3)	23	11	22	10	(3)	01	12	21	32	(3)	01	21	12	32
805.					806.					807.					808.				
21.*c*′	00	22	11	33	21′*c*	00	11	22	33	22.*d*′	30	31	02	03	22′*d*	30	02	31	03
f 801.	31	30	03	02		12	32	01	21		22	00	33	11		13	10	23	20
	12	01	32	21		31	03	30	02		13	23	10	20		22	33	00	11
(3)	23	13	20	10	(3)	23	20	13	10	(3)	01	12	21	32	(3)	01	21	12	32
809.					810.					811.					812.				
21′*c*	00	13	20	33	21.*c*′	00	20	13	33	22′*d*	30	12	21	03	22.*d*′	30	21	12	03
f 801.	12	30	03	21		31	32	01	02		13	00	33	20		22	10	23	11
	31	01	32	02		12	03	30	21		22	23	10	11		13	33	00	20
(3)	23	22	11	10	(3)	23	11	22	10	(3)	01	31	02	32	(3)	01	02	31	32
813.					814.					815.					816.				
19′*a*	00	22	11	33	19.*a*′	00	11	22	33	20′*b*	30	12	21	03	20.*b*′	30	21	12	03
f 801.	12	30	03	21		31	32	01	02		22	00	33	11		13	10	23	20
	31	01	32	02		12	03	30	21		13	23	10	20	8	22	33	00	11
(3)	23	13	20	10	(3)	23	20	13	10	(3)	01	31	02	32	(3)	01	02	31	32

817.					818.					819.					820.				
23.*b*′	12	01	32	21	23′*b*	12	32	01	21	24.*a*′	30	31	02	03	24′*a*	30	02	31	03
f 801.	31	30	03	02		00	20	13	33		01	12	21	32		22	10	23	11
	00	13	20	33		31	03	30	02		22	23	10	11		01	21	12	32
(3)	23	22	11	10	(3)	23	11	22	10	(3)	13	00	33	20	(3)	13	33	00	20
821.					822.					823.					824.				
25.*d*	12	22	11	21	25′*d*′	12	11	22	21	26.*c*	30	31	02	03	26′*c*′	30	02	31	03
f 817.	31	30	03	02		00	20	13	33		22	12	21	11		01	10	23	32
	00	13	20	33		31	03	30	02		01	23	10	32		22	21	12	11
(3)	23	01	32	10	(3)	23	32	01	10	(3)	13	00	33	20	(3)	13	33	00	20
825.					826.					827.					828.				
25′*d*′	12	01	32	21	25.*d*	12	32	01	21	26′*c*′	30	00	33	03	26.*c*	30	33	00	03
f 817.	00	30	03	33		31	20	13	02		01	12	21	32		22	10	23	11
	31	13	20	02		00	03	30	33		22	23	10	11		01	21	12	32
(3)	23	22	11	10	(3)	23	11	22	10	(3)	13	31	02	20	(3)	13	02	31	20
829.					830.					831.					832.				
23′*b*	12	22	11	21	23.*b*′	12	11	22	21	24′*a*	30	00	33	03	24.*a*′	30	33	00	03
f 817.	00	30	03	33		31	20	13	02		22	12	21	11		01	10	23	32
	31	13	20	02		00	03	30	33		01	23	10	32		22	21	12	11
(3)	23	01	32	10	(3)	23	32	01	10	(3)	13	31	02	20	(3)	13	02	31	20

VII. 21. (no reversals)

Solutions 801–816: Φ_4 Solutions 817–832: Φ_5

Category Three

						U					S					US			
833.					834.					835.					836.				
1.e	00	01	32	33	1′e′	00	32	01	33	2.f	31	30	03	02	2′f′	31	03	30	02
Lead	30	31	02	03		23	21	12	10		01	00	33	32		22	20	13	11
	23	12	21	10		30	02	31	03		22	13	20	11		01	33	00	32
(3)	13	22	11	20	(3)	13	11	22	20	(3)	12	23	10	21	(3)	12	10	23	21
837.					838.					839.					840.				
3.g	00	22	11	33	3′g′	00	11	22	33	4.h	31	30	03	02	4′h′	31	03	30	02
f 833.	30	31	02	03		23	21	12	10		22	00	33	11		01	20	13	32
	23	12	21	10		30	02	31	03		01	13	20	32		22	33	00	11
(3)	13	01	32	20	(3)	13	32	01	20	(3)	12	23	10	21	(3)	12	10	23	21
841.					842.					843.					844.				
5.g′	00	01	32	33	5′g	00	32	01	33	6.h′	31	23	10	02	6′h	31	10	23	02
f 833.	23	31	02	10		30	21	12	03		01	00	33	32		22	20	13	11
	30	12	21	03		23	02	31	10		22	13	20	11		01	33	00	32
(3)	13	22	11	20	(3)	13	11	22	20	(3)	12	30	03	21	(3)	12	03	30	21
845.					846.					847.					848.				
7.e′	00	22	11	33	7′e	00	11	22	33	8.f′	31	23	10	02	8.f	31	10	23	02
f 833.	23	31	02	10		30	21	12	03		22	00	33	11		01	20	13	32
	30	12	21	03		23	02	31	10		01	13	20	32		22	33	00	11
(3)	13	01	32	20	(3)	13	32	01	20	(3)	12	30	03	21	(3)	12	03	30	21

849.					850.					851.					852.				
1.f	01	00	33	32	1′f′	01	33	00	32	2.e	30	31	02	03	2′e′	30	02	31	03
f 833.	31	30	03	02		22	20	13	11		00	01	32	33		23	21	12	10
	22	13	20	11		31	03	30	02		23	12	21	10		00	32	01	33
(3)	12	23	10	21	(3)	12	10	23	21	(3)	13	22	11	20	(3)	13	11	22	20
853.					854.					855.					856.				
3.h′	01	23	10	32	3′h	01	10	23	32	4.g′	30	31	02	03	4′g	30	02	31	03
f 849.	31	30	03	02		22	20	13	11		23	01	32	10		00	21	12	33
	22	13	20	11		31	03	30	02		00	12	21	33		23	32	01	10
(3)	12	00	33	21	(3)	12	33	00	21	(3)	13	22	11	20	(3)	13	11	22	20
857.					858.					859.					860.				
5.h	01	00	33	32	5′h′	01	33	00	32	6.g	30	22	11	03	6′g′	30	11	22	03
f 849.	22	30	03	11		31	20	13	02		00	01	32	33		23	21	12	10
	31	13	20	02		22	03	30	11		23	12	21	10		00	32	01	33
(3)	12	23	10	21	(3)	12	10	23	21	(3)	13	31	02	20	(3)	13	02	31	20
861.					862.					863.					864.				
7.f′	01	23	10	32	7′f	01	10	23	32	8.e′	30	22	11	03	8′e	30	11	22	03
f 849.	22	30	03	11		31	20	13	02		23	01	32	10		00	21	12	33
	31	13	20	02		22	03	30	11		00	12	21	33		23	32	01	10
(3)	12	00	33	21	(3)	12	33	00	21	(3)	13	31	02	20	(3)	13	02	31	20

VII. 22. (no reversals)

Solutions 833–848: Φ_6 Solutions 849–864: Φ_7

Category Three

							U					S					US		
865.					866.					867.					868.				
11.*a*′	00	33	20	13	11′*a*	00	20	33	13	12.*b*′	10	31	22	03	12′*b*	10	22	31	0
Lead	31	10	03	22		12	32	21	01		33	00	13	20		02	30	23	1
	12	21	32	01		31	03	10	22		02	23	30	11		33	13	00	2
(11)	23	02	11	30	(12)	23	11	02	30	(11)	21	12	01	32	(12)	21	01	12	3
869.					870.					871.					872.				
13.*a*′	33	00	13	20	13′*a*	33	13	00	20	14.*b*′	23	02	11	30	14′*b*	23	11	02	3
c 865.	02	23	30	11		21	01	12	32		00	33	20	13		31	03	10	2
	21	12	01	32		02	30	23	11		31	10	03	22		00	20	33	1
(11)	10	31	22	03	(12)	10	22	31	03	(11)	12	21	32	01	(12)	12	32	21	0

873.					874.					875.					876.				
1′*j*	00	33	01	32	1.*j*′	00	01	33	32	2′*j*	20	23	11	12	2.*j*′	20	11	23	1
Lead	23	20	12	11		30	31	03	02		33	00	32	01		10	21	13	2
	30	03	31	02		23	12	20	11		10	13	21	22		33	32	00	0
(11)	13	10	22	21	(12)	13	22	10	21	(11)	03	30	02	31	(12)	03	02	30	3
877.					878.					879.					880.				
1′*j*	33	00	32	01	1.*j*′	33	32	00	01	2′*j*	13	10	22	21	2.*j*′	13	22	10	2
c 873.	10	13	21	22		03	02	30	31		00	33	01	32		23	12	20	1
	03	30	02	31		10	21	13	22		23	20	12	11		00	01	33	3
(11)	20	23	11	12	(12)	20	11	23	12	(11)	30	03	31	02	(12)	30	31	03	0

VII. 23. (no reversals)

Solutions 865–868: Ω_1 Solutions 869–872: Ω_2

Solutions 873–876: Ω_3 Solutions 877–880: Ω_4

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Albrecht Dürer's 'Melancholia' (The British Museum). Note the four-by-four magic square in the upper right-hand corner in which the date 1514 appears in the two middle cells of the bottom row.